Math 280: 11.3 Notes on Remainder, Error and Estimating the sum of a series, *s*
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Example: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Since it is a p-series with p>1, we know it converges. We also know that the function $f(x) = \frac{1}{x^4}$ is continuous, positive and decreasing on $[1,\infty)$

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1. Estimate the sum of the series, *s*, using the 10[th] partial sum s_{10}.

$$s_{10} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots + \frac{1}{10^4} \approx 1.082037$$

on calculator: sum(seq(1/x^4, x, 1, 10))

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2. How good is this estimate of *s* ?
Let's calculate the upper bound of the remainder (also called error). The error/remainder to this estimate s_{10} is the sum of all the terms $s_{11} + s_{12} + s_{13} + s_{14} + \ldots \quad = R_{10}$

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty} \int_{10}^{t} \frac{1}{x^4}dx = \lim_{t\to\infty} \left. \frac{-1}{3x^3} \right|_{10}^{t} = \lim_{t\to\infty} \left(\frac{-1}{3t^3} + \frac{1}{3(10)^3} \right) = \frac{1}{3(10)^3} = \frac{1}{3000} = .000333$$

So the error is $\le .000333$.

Note: In general, $R_n \le \frac{1}{3(n)^3}$ for any integer *n*

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3. Improve this estimate of *s* for with the formula:

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^4}dx \quad \le \quad s \quad \le \quad s_{10} + \int_{10}^{\infty} \frac{1}{x^4}dx$$

So:

$$s_{10} + \frac{1}{3(11)^3} \quad \le \quad s \quad \le \quad s_{10} + \frac{1}{3(10)^3}$$

$$1.082037 + .000250 \quad \le \quad s \quad \le \quad 1.082037 + .000333$$

$$1.082287 \quad \le \quad s \quad \le \quad 1.082370$$

We can approximate *s* with the midpoint of this interval so:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = s \approx \frac{1.082287 + 1.082370}{2} = 1.0823285$$

And the error of this estimate is at most half the length of the interval so:

$$\text{error} < \frac{1.082370 - 1.082287}{2} = .0000415$$

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4. Find a value of *n* that will ensure that the error in the approximation of $s \approx s_n$ is less than .001.

So find *n* so that $.001 = R_n > \frac{1}{3(n)^3}$ We need to solve for *n* in: $.001 > \frac{1}{3n^3}$

so $n^3 > 333.33$ which means $n > 6.93$

This means that *n*=7 would give an estimate s_7 with an error < .001