Math 280: 11.4 Convergence Tests: The Comparison Test and The Limit Comparison Test
Vanden Eynden

The Comparison Test: If $\sum a_n$ and $\sum b_n$ are series with positive terms and

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Use the Comparison Test to determine if each series converges or diverges.

D 1. $\sum_{n=1}^{\infty} \frac{3}{n-4}$ $\quad \frac{3}{n-4} > \frac{3}{n}$

$\sum \frac{3}{n}$ diverges by p-series, p=1

So $\sum \frac{3}{n-4}$ diverges by Comp. test. (C.T.)

C 2. $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ $\quad \frac{1}{3^n+1} < \frac{1}{3^n}$

$\sum \frac{1}{3^n}$ converges by geometric series with $r = \frac{1}{3} < 1$.

So by C.T., $\sum \frac{1}{3^n+1}$ converges.

C 3. $\sum_{n=1}^{\infty} \frac{2}{n^4+n}$ $\quad \frac{2}{n^4+n} < \frac{2}{n^4}$

$\sum \frac{2}{n^4}$ converges by p-series, p=4

So $\sum \frac{2}{n^4+n}$ converges by C.T.

C 4. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ $\quad \frac{\cos^2 n}{n^2+1} \le \frac{1}{n^2+1} < \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges by p-series, p=2

So $\sum \frac{\cos^2 n}{n^2+1}$ converges by C.T.

D 5. $\sum_{n=2}^{\infty} \frac{n+1}{n^2-n}$ $\quad \frac{n+1}{n^2-n} > \frac{n}{n^2} = \frac{1}{n}$

$\sum \frac{1}{n}$ diverges by p-series, p=1

So $\sum \frac{n+1}{n^2-n}$ diverges by C.T.

C 6. $\sum_{n=1}^{\infty} \frac{2^n-1}{3^n+2n}$ $\quad \frac{2^n-1}{3^n+2n} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

$\sum \left(\frac{2}{3}\right)^n$ converges by geometric series with $r = \frac{2}{3} < 1$

So $\sum \frac{2^n-1}{3^n+2n}$ converges by C.T.

7. $\sum_{n=1}^{\infty} \frac{5\sin^2 n}{n!}$ $\quad \frac{5\sin^2 n}{n!} \le \frac{5}{n!} \le \frac{5}{2^{n-1}}$

$\sum \frac{5}{2^{n-1}}$ converges by geometric series with $r = \frac{1}{2} < 1$

So $\sum \frac{5\sin^2 n}{n!}$ converges by C.T.

The Limit Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.

Use the limit comparison test to show whether the series converges or diverges.

D 1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

$\boxed{\text{compare to } \frac{1}{\sqrt{n}}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+4}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+4}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+4}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{4}{n}}} = 1 > 0.$$

Since $\sum \frac{1}{\sqrt{n}}$ diverges by p-series, $p=\frac{1}{2}<1$, $\sum \frac{1}{\sqrt{n+4}}$ diverges by Limit Comparison test

---

C 2. $\sum_{n=1}^{\infty} \frac{1}{e^n - 5}$

$\boxed{\text{compare to } \frac{1}{e^n}}$

$$\lim_{n\to\infty} \frac{1}{e^n-5} \cdot \frac{e^n}{1} = \lim_{n\to\infty} \frac{e^n}{e^n-5} = \lim_{n\to\infty} \frac{1}{1-\frac{5}{e^n}} = 1 > 0$$

Since $\sum \frac{1}{e^n}$ converges by geometric series w/ $r=\frac{1}{e}<1$, $\sum \frac{1}{e^n-5}$ converges by L.C.T.

D 3. $\sum_{n=1}^{\infty} \frac{1}{\ln n + n}$

$\boxed{\text{compare to } \frac{1}{n}}$

$$\lim_{n\to\infty} \frac{1}{\ln n+n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{\ln n + n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}+1} = 1 > 0$$

Since $\frac{1}{n}$ diverges as the harmonic p-series, $p=1$, $\sum \frac{1}{\ln n + n}$ also diverges by L.C.T.