Math 280: 11.7 Strategy for Testing Series

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ DIVERGES By Divergence test

$$\lim_{n\to\infty} \frac{2^n}{n^3} \overset{H}{=} \lim_{n\to\infty} \frac{2^n \ln 2}{3n^2} \overset{H}{=} \lim_{n\to\infty} \frac{2^n(\ln 2)^2}{6n} \overset{H}{=} \lim_{n\to\infty} \frac{2^n(\ln 2)^3}{6} = \infty$$
DNE.

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2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n+3}$ ① Alternating ✓

② $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n+3} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ ✓

Converges (conditionally) by alternating series test.

③ check decreasing: $f(x) = \frac{\ln x}{x+3}$

$f'(x) = \frac{(x+3)\frac{1}{x} - (\ln x)(1)}{(x+3)^2}$ check: $1 + \frac{3}{x} - \ln x < 0$ for $x > 4.97$ (used calculator)

So decreasing for $n \geq 5$ ✓

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3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}}$ DIVERGES by p-series, $p = \frac{4}{5} < 1$.

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4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+6n}}$ DIVERGES by limit comparison to $b_n = \frac{1}{\sqrt[3]{n^2}}$.

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2+6n}} \cdot \frac{\sqrt[3]{n^2}}{1} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2+6n}} = \sqrt[3]{\lim_{n\to\infty} \frac{1}{1+\frac{6}{n}}} = 1 > 0$$

Since $\sum \frac{1}{\sqrt[3]{n^2}}$ diverges by p-series, $p = 2/3 < 1$, So does original series

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5. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n!} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=1}^{\infty} \frac{1}{2^n}$ Check each series separately.

① $\sum \frac{1}{n!}$ converges by comparison test, since $\frac{1}{n!} < \frac{1}{2^{n-1}}$ geometric with $r = \frac{1}{2} < 1$.

② $\sum \frac{1}{2^n}$ converges by geometric with $r = \frac{1}{2} < 1$

③ So original converges.

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6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+e)(n+\pi)} = \sum \frac{n}{n^2 + en + \pi n + e\pi}$ compare to $\frac{n}{n^2} = \frac{1}{n}$.

$$\lim_{n\to\infty} \left(\frac{n}{n^2 + (e+\pi)n + e\pi} \cdot \frac{n}{1}\right) = \lim_{n\to\infty} \frac{n^2}{n^2 + (e+\pi)n + e\pi} = 1 > 0$$

So by limit comparison test, since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{n}{(n+e)(n+\pi)}$ diverges.

7. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

(terms positive, try integral test.)

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty}\int_1^t x^2 e^{-x^3} = \overset{\text{u-sub.}}{\begin{Bmatrix} u=-x^3 \\ du=-3x^2\,dx \end{Bmatrix}} = \lim_{t\to\infty}\int_1^t -\frac{1}{3}e^u\,du$$

$$= \lim_{t\to\infty} -\frac{1}{3}e^{-x^3}\Big|_1^t = \lim_{t\to\infty}\left(-\frac{1}{3}e^{-t^3} - -\frac{1}{3}e^{-1}\right) = \frac{1}{3e}$$

So $\sum n^2 e^{-n^3}$ converges by the integral test.

(could also use ratio test)

8. $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ covergees absolutely by root test.

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{n^2}\right)^n} = \lim_{n\to\infty}\frac{2n+1}{n^2} = 0 < 1$$

9. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$ check $\sum\left|\frac{\sin 2n}{1+2^n}\right|$. Since $\frac{|\sin 2n|}{1+2^n} \leq \frac{1}{1+2^n} < \frac{1}{2^n}$

$\sum \frac{|\sin 2n|}{1+2^n}$ converges by comparison test to $\sum \frac{1}{2^n}$ (geometric $r=\frac{1}{2}<1$)

So original $\boxed{\text{converges absolutely}}$

10. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty}\left(\frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{(n+1)n!}{e\cdot e^n}\cdot\frac{e^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{n+1}{e}\right) = \infty$$

DIVERGES by the ratio test.

11. $\sum_{n=1}^{\infty} \frac{1}{2+\cos n}$ Diverges by divergence test.

$\lim_{n\to\infty}\frac{1}{2+\cos n}$ = DNE, Since $\frac{1}{2+\cos n}$ oscillates between $\frac{1}{3}$ and 1.

12. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}$

$$\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)(2(n+1)-1)}{2\cdot 5\cdot 8\cdots(3n-1)(3(n+1)-1)}\cdot\frac{2\cdot 5\cdot 8\cdots(3n-1)}{1\cdot 3\cdot 5\cdots(2n-1)} = \lim_{n\to\infty}\frac{2n+2-1}{3n+3-1}$$

$$= \lim_{n\to\infty}\frac{2n+1}{3n+2} = \frac{2}{3} < 1$$

Converges absolutely by ratio test