SEQUENCES

Theorem 3:

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Squeeze Theorem for Sequences:

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

Theorem 6:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 7:

If $\lim_{n\to\infty} a_n = L$ and the function *f* is continuous at *L*, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Geometric sequences:

The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of *r*.

$$\lim_{n\to\infty}\{r^n\} = \begin{cases} 0 \ \textit{if} & -1 < r < 1 \\ 1 \ \textit{if} & r = 1 \end{cases}$$

Monotonic Sequence Theorem:

Every bounded, monotonic sequence is convergent.

(A) Definition Partial Sums:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum,

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence $\{s_n\}$, the sequence of partial sums $\{s_1, s_2, s_3, \ldots\}$ is convergent and its limit is a real number, $\lim_{n\to\infty} s_n = s$, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series. Otherwise, the series is called divergent.

(B) Geometric Series: The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$ is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, the series is divergent.

(C) Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(D) Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words,

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(E) P-series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(F) Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$.

(G) Series Sum Estimate for the Integral Test: $s_n + \int_{n+1}^{\infty} f(x)dx \le s \le s_n + \int_{n}^{\infty} f(x)dx$

The midpoint of this interval is an estimate of *s*, with error < (half the interval's length).

(H) The Comparison Test: If $\sum a_n$ and $\sum b_n$ are series with positive terms and

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

(I) The Limit Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and $c > 0$, then either both series converge or both diverge.