Math 280 Final Exam Study Guide

FINAL EXAM
Wednesday, Dec 9
1:45 – 3:45pm
Room 331

7.1 Integration by parts: $\int u\,dv = u \cdot v - \int v\,du$

L – Logs	
I – Inverse Trig	Higher on the list: u
A – Algebraic	
T – Trig	Lower on the list: dv
E – Exponential	

7.2 Trigonometric Integrals

If $\int \sin^m x \cos^n x\,dx$, use u = sinx if n odd, u = cosx if m odd, or if both even powers, half angle formula

If $\int \tan^m x \sec^n x\,dx$, use u = tanx if n even, u = secx if m odd

7.3 Trigonometric Substitution

Expression in the integrand	*Substitution*
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{x^2 + a^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

7.4 Partial Fractions

7.5 Strategy for Integration

7.7 Approximate Integrals: Midpoint, Trapezoidal, Simpson's Rule

7.8 Improper Integrals: $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$ infinite interval or

$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$ infinite discontinuity at endpoint

Chapter 11 Sequences and Series!!!

a. What is a sequence? What does it take for a sequence to converge?

b. What is a series? What does it take for a sequence to converge?

c. How is a series different from a sequence?

d. Your "**SENSE**" about a series' convergence/divergence VS. **PROVING** convergence/divergence

e. **Divergence Test**

f. **Geometric Series** and their sums

g. **Telescoping Series** and their sums

h. **p-series**

i. **Integral Test**: If integral converges, so does series

j. **Remainder estimate** for the integral test

k. **Direct Comparison Test**

l. **Limit Comparison Test**

m. **Alternating Series Test**: Check conditions! 1. Limit is 0, and 2. Decreasing function (find derivative)

n. **Remainder estimate** for alternating series

o. **Absolute Convergence, Conditional Convergence, Divergence**: Can't use Alt. Series Test to prove Absolute Convergence!

p. **Ratio and Root Tests**: If L<1, converge. If L>1, diverge. Use with products of factorials, exponentials These tests don't work on p-series, rational or algebraic functions of n.

q. **Power Series centered at x = a**

r. **Radius of convergence and Interval of convergence**

s. **Geometric Power Series representations of functions of the form** $\frac{1}{1-X}$

t. **Differentiation and Integration of power series**

u. **Taylor and MacLaurin Series**

v. **Binomial Series**

10.1 Parametric Equations

10.2 Calculus of Parametric Equations: Tangents, Concavity, Arc length

10.3 Polar Coordinates, Graphs and Equations

10.4 Calculus in Polar Coordinates: Areas, points of intersections

10.6 Conic Sections in Polar Coordinates $r = \frac{ed}{1 \pm e\cos\theta}$ *or* $r = \frac{ed}{1 \pm e\sin\theta$}

The conic is:

a) an ellipse if $e < 1$

b) a parabola if $e = 1$

c) a hyperbola if $e > 1$

Sample Practice Problems:

1. Evaluate the integrals.

a. $\int_{\sqrt{e}}^{e} \frac{\ln x}{x^2} dx$

b. $\int_0^4 x^3\sqrt{16-x^2}\,dx$

c. $\int \frac{1}{x^2\left(4-x^2\right)} dx$

d. $\int_0^{\infty} xe^{-x} dx$

e. $\int \ln(x+2)\ dx$

f. $\int_2^{\infty} \frac{dx}{x \ln x}$

g. $\int_0^2 \frac{2x}{x^2-7x+12}\ \ dx$

h. $\int x^5 \ln x$

2. For each of the following integrals, name one integration technique you could use to evaluate the integral effectively. DO NOT INTEGRATE!!

a. $\int \frac{dx}{\left(1-x^2\right)^{3/2}}$

b. $\int x^2 e^{x^3} dx$

c. $\int \frac{dx}{x^2-4x-5}$

3. *Let* $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n (x+2)^n}{n2^n}=\frac{-1}{1\cdot 2}(x+2)+\frac{1}{2\cdot 2^2}(x+2)^2-\frac{1}{3\cdot 2^3}(x+2)^3+\cdots$

a. Find the radius of convergence and interval of convergence of the power series representing the function f(x).

b. Estimate $f(1)$ using the first four terms of the power series.

c. Estimate $f(-1)$ using the first four terms of the power series.

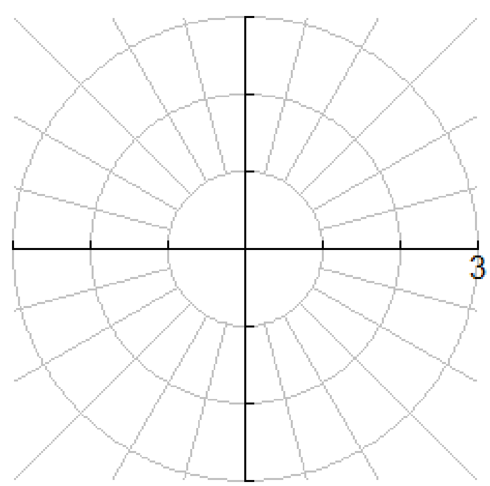
d. Which estimate $f(1)$ or $f(-1)$ is better? Explain.

4. Consider the function described by $r=1-2\cos\theta$.

A. Sketch the graph of the polar equation.

B. Find the slope of the tangent line to the polar curve when $\theta=\frac{\pi}{2}$.

C. Finding the area between the inner and outer loops



5. Determine whether the series converges or diverges. It the series is geometric and converges, find the sum. Work on problems in section 11.7 in your textbook.

a. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ b. $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n(n+3)}$ c. $\sum_{n=1}^{\infty} \frac{n}{3n+2}$

d. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln(n))^2}$ e. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n}$ f. $\sum_{n=1}^{\infty} \frac{5}{n^2-10}$

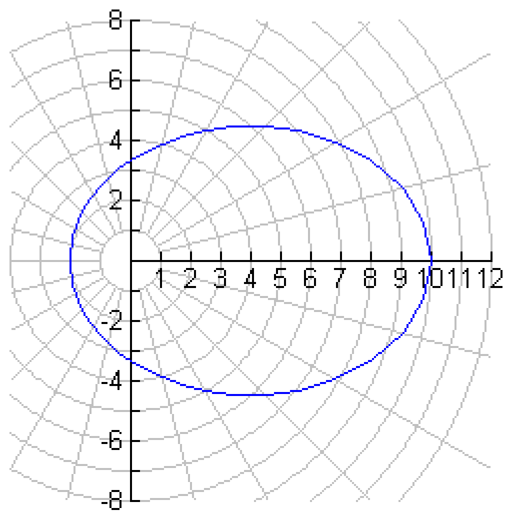
g. $\sum_{n=1}^{\infty} \frac{5^{n+1}}{\sqrt{3}^n}$ h. $\sum_{n=1}^{\infty} \frac{n^2 3^n}{n!}$

6. Use the integral test to determine convergence or divergence. $\sum_{n=1}^{\infty} \frac{1}{(4n+1)^5}$

7. Find the Maclaurin series representation for the given function. Specify the interval of convergence.

$$f(x) = \frac{1}{1-4x}$$

8. Given the polar equation $r = \frac{10}{3-2\cos\theta}$ and its graph below:



a. Find $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}}$

b. The integral $\int_{\pi/6}^{\pi/3} \frac{1}{2}\left(\frac{10}{3-2\cos\theta}\right)^2 d\theta$ is used to evaluate the area of a sector within the shape to the right. Draw and shade the sector on the graph and estimate the integral by estimating the area contained within the sector.

c. Find the following for this ellipse: eccentricity, directrix, exact y-intercepts (in polar coordinates)

9. Given the Maclaurin series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $(-\infty, \infty)$, derive the Maclauren series for $f(x) = xe^{-x}$.

Indicate the radius of convergence for the new series.

10. Consider the infinite series given by $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1) \cdot 3^n}{2^{2n+1}}$

A. Determine whether the series converges absolutely, converges conditionally, or diverges.

B, Does the series satisfy the conditions of the alternating series test? How many terms must you use so that your error is less than 0.00001?

11. Given the general formula for a Taylor polynomial $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$:

(a) Find a third degree Taylor polynomial, $T_3(x)$ for $\cos x$ centered at $\frac{\pi}{3}$.

(b) Estimate the value of $\cos(70^\circ) = \cos\left(\frac{70^\circ \pi}{180^\circ}\right)$ using the Taylor polynomial, $T_3(x)$.

12. Find the interval of convergence for the power series: $\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+2}$

13. Find the equation of the tangent line to the curve $x = t^2 + 4t + 1, \quad y = 2t - t^2$ at $t = -1$

14. Sketch the parametric curve $x = 1 + e^{2t},\ y = e^t$ and eliminate the parameter to find the Cartesian (in x and y) equation of the curve. Is the curve concave up or concave down at (2, 1)?

15. Find dy/dx and d^2y/dx^2 for $x = 1 + t^2,\ y = t - t^3$. At what points does the curve has horizontal or vertical tangents? Set up an integral that represents the length of the curve <u>once</u> around the loop. Use your calculator to estimate the length of the loop.

16. Sketch the polar graphs and find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$

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17. Decide whether the following represent a parabola, hyperbola, or ellipse:

a. $r = \frac{12}{3+\sin\theta}$ b. $r = \frac{1}{1-2\sin\theta}$ c. $r = \frac{5}{3-3\cos\theta}$

18. Write a polar equation of a conic with the focus at the origin and the given data:

a. Parabola, vertex at $(3, \pi)$

b. Ellipse, eccentricity ½, directrix $y = 3$

<u>Answers</u> to Practice Problems from

Math 280 Final Exam Study Guide

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1. a. $\frac{3}{2\sqrt{e}} - \frac{2}{e}$ (by parts) b. 2048/15 (trig sub)

   c. $-\frac{1}{4x} - \frac{1}{16}\ln|2-x| + \frac{1}{16}\ln|x+2| + C$ (partial fraction decomp)

   d. 1 (improper int., by parts) e. $x\ln(x+2) + 2\ln|x+2| - x + C$ (by parts)

   f. diverges (improper int., u-sub)

   g. $8\ln 2 - 8\ln 4 + 6\ln 3 = \ln\left(\frac{729}{256}\right)$ (partial fraction decomp)

   h. $\frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C$ (by parts)

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2. a. trig substitution, $x = \sin\theta$ b. u-substitution c. partial fraction decomp

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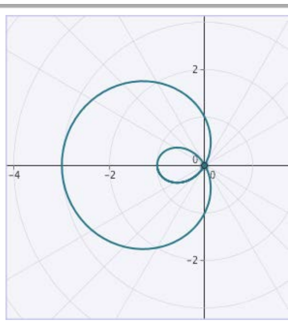
3.
   a. R = 2 , Interval: (–4, 0] b. –0.234575 c. –0.682

   d. $f(-1)$, since $x = -1$ is within the radius of convergence. (This means $f(1)$ diverges)

4.a. 

b. $\frac{dy}{dx} = -2$

c. $= 2\int_{\pi/3}^{\pi} \frac{1}{2}(1-2\cos\theta)^2 d\theta - 2\int_{0}^{\pi/3} \frac{1}{2}(1-2\cos\theta)^2 d\theta = \pi + 3\sqrt{3}$

5. a. Converges (Ratio Test)

   b. Converges (Alternating Series Test); Converges conditionally (limit comp. test)

   c. Diverges (Divergence Test)

   d. Converges (Alternating Series Test) ; Converges absolutely (integral test)

   e. Converges (Comparison test)

   f. Converges (Limit comp. test)

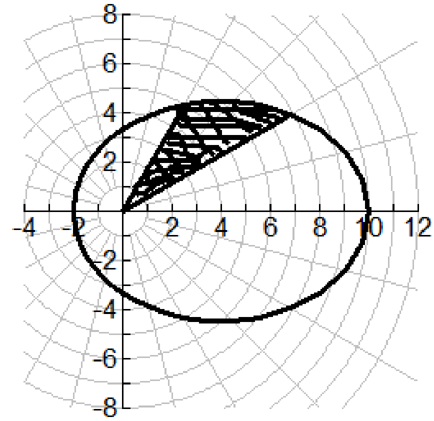
   g. Diverges (Geometric Series with r > 1)

   h. Converges (Ratio Test)

6. Converges by integral test, since $\int_1^{\infty} \frac{1}{(4x+1)^5} dx = \frac{1}{10000}$

7. $\sum_{n=0}^{\infty} 4^n x^n$, for $|x| < \frac{1}{4}$. Interval: $\left(-\frac{1}{4}, \frac{1}{4}\right)$

8. This problem is super messy!



a. $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4}{3} - \sqrt{3}$

b. My Area estimate $\approx \frac{\pi}{6} \cdot \frac{(6.2)^2}{2} \approx 10.1$

WolframAlpha integral estimate = 10.7538

c. e = 2/3; directrix: $x = -5$; y-intercepts: $\left(\frac{10}{3}, \frac{\pi}{2}\right)$ and $\left(\frac{10}{3}, \frac{3\pi}{2}\right)$

9. $xe^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!}$ R= ∞

10. a. Absolutely convergent by Ratio Test.

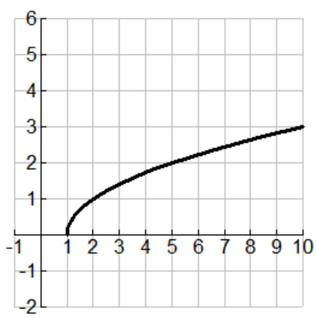
b. Yes (alternating, decreasing (check derivative), limit goes to 0)
Need 52 terms (n=0 up to and including n=51)

11. a. $T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4\sqrt{3}}\left(x - \frac{\pi}{3}\right)^3$

b. $\cos(70^\circ) \approx T_3\left(\frac{7\pi}{18}\right) \approx 0.3420$

12. [3,5]

13. $y = 2x + 1$

14. 

Cartesian: $x = 1 + y^2, y > 0$.

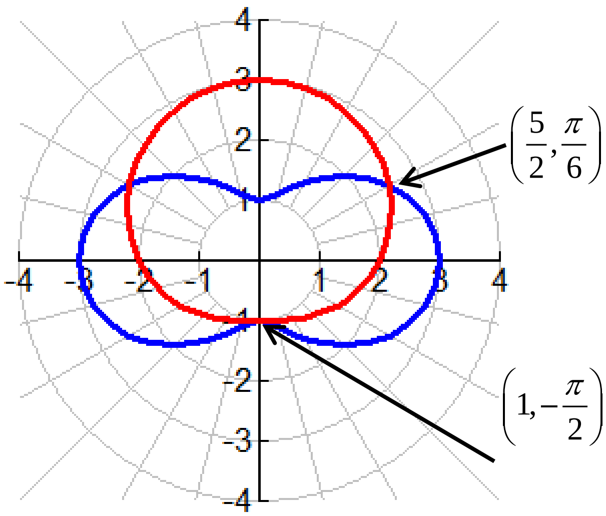
$\frac{d^2y}{dx^2} = \frac{-1}{4e^{3t}}$, which is < 0 for all t. Concave down at t = 0.

15. $\frac{dy}{dx} = \frac{1-3t^2}{2t}$, $\frac{dy}{dx} = \frac{-1-3t^2}{4t^3}$;

vertical tangent at (1,0) when t = 0; horizontal tangent at $\left(\frac{4}{3}, \pm\frac{2}{3\sqrt{3}}\right)$ when t = $\pm\frac{1}{\sqrt{3}}$;

$\int_{-1}^{1} \sqrt{(2t)^2 + (1-3t^2)^2}\ dt = \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1} = 2.71559$

16. $A = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[\left(2+\cos 2\theta\right)^2 - \left(2+\sin\theta\right)^2\right] d\theta = \frac{51}{16}\sqrt{3}$



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17. a ellipse, e=1/3, d=12, directrix: y = 12

b. hyperbola, e=2, d=1/2, directrix: y = -1/2

c. parabola, e=1, d=5/3, directrix: x = -5/3

18. a. $r = \frac{6}{1-\cos\theta}$

b. $r = \frac{3}{2+\sin\theta}$

Math 280 FINAL EXAM Formula/Theorem Sheet

Vanden Eynden **(You will be provided a fresh copy on exam day)**

Derivatives of Inverse Trig Functions:

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

Established Integration Formulas

$$\int \tan x\,dx = \ln|\sec x| + C$$
$$\int \cot x\,dx = \ln|\sin x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

Half-Angle Formulas

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$
$$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

Double-Angle Formulas

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Product Formulas

$$\sin A\cos B = \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]$$
$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$
$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Trigonometric Substitution

Expression in the integrand	*Substitution*
$\sqrt{a^2-x^2}$	$x = a\sin\theta$
$\sqrt{x^2+a^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

(A) Definition Partial Sums:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum,

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence $\{s_n\}$, the sequence of partial sums $\{s_1, s_2, s_3, \ldots\}$ is convergent and its limit is a real number, $\lim_{n\to\infty} s_n = s$, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series. Otherwise, the series is called divergent.

(B) Geometric Series: The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$ is convergent if $|r| < 1$ and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. If $|r| \geq 1$, the series is divergent.

(C) Test for Divergence: If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(D) Integral Test: Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words,

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(E) P-series: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

(F) Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = S - S_n$, then $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)dx$.

(G) Series Sum Estimate for the Integral Test: $s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx$

The midpoint of this interval is an estimate of *s*, with error < (half the interval's length).

(H) The Comparison Test: If $\sum a_n$ and $\sum b_n$ are series with positive terms and

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

(I) The Limit Comparison test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where *c* is a finite number and $c > 0$, then either both series converge or both diverge.

(J) The Alternating Series Test:

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ...$ where $b_n > 0$ satisfies

(i) $b_{n+1} \le b_n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series converges.

(K) Alternating Series Estimation Theorem:

If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $0 \le b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

Then $|R_n| = |s - s_n| \le b_{n+1}$

(L) Absolute Convergence:

If $\sum |a_n|$ converges, then $\sum a_n$ converges (absolutely).

(M) The Ratio Test for Absolute Convergence:

Let $\sum a_n$ be a series with non-zero terms and suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$;

i. If L < 1, the series $\sum a_n$ is absolutely convergent.

ii. If L > 1 or $L = \infty$, then the series $\sum a_n$ diverges.

iii. If L = 1 , the test is inconclusive. The series may be convergent or divergent. Use another test (not The Root Test)

(N) The Root Test for Absolute Convergence:

Suppose $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$

i. If L < 1, then the series $\sum a_n$ is absolutely convergent.

ii. If L > 1 or $L = \infty$, then the series $\sum a_n$ diverges .

iii. If L = 1 , the test is inconclusive. The series may be convergent or divergent. Use another test (not The Ratio Test)

Taylor Series Expansion of $f(x)$ at $x = a$ (or "about *a*", or "centered at *a*")

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Series Expansion of $f(x)$ at $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

First Derivative of parametric equations: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ if $\dfrac{dx}{dt} \neq 0$

Second Derivative of parametric equations: $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ if $\dfrac{dx}{dt} \neq 0$

Arc Length for parametric equations, for $\alpha < t < \beta$: $L = \displaystyle\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

First Derivative of polar equations: $\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

Area of polar region: $A = \displaystyle\int_a^b \frac{1}{2}\left[f(\theta)\right]^2\ d\theta$

Conic Sections in Polar Coordinates

$$r = \frac{ed}{1 \pm e\cos\theta} \quad or \quad r = \frac{ed}{1 \pm e\sin\theta}$$

The conic is:

d) an ellipse if $e < 1$

e) a parabola if $e = 1$

f) a hyperbola if $e > 1$