Supplement to: *A discrete, energetic approach to rocket propulsion*

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Three discrete propulsion models compared to continuous thrust: Constant energy per unit mass, constant impulse, and constant relative velocity

As discussed in section 1 of the main article, previous treatments of rocket propulsion, going back in time at least as far as Tsiolkovsky's 1903 derivation of continuous thrust,* have made different assumptions about the speeds of the expelled exhaust. Here I summarise and compare three models of discrete propulsion that exist in the literature, and attempt to reduce the confusion by tabulating and plotting their functional forms in one place.

For discrete propulsion with the assumption of constant *mechanical energy* per unit mass α, the rocket's total speed change Δv is given by equation (8) in the main article, where ϕ is the fraction of the rocket's total mass in the form of fuel. As shown in section 3.2, equation (10) can be derived by assuming that each pellet provides a constant *impulse* per unit mass (with a center-of-mass pellet speed of $v_{\mathrm{p}} = \sqrt{2\alpha}$.) Finally, to find the Δv for a model where each pellet is expelled with a constant *relative velocity* with respect to the rocket, rearrange equation (6),

$$\Delta v = \frac{m_{\mathrm{p}} v_{\mathrm{rel}}}{m + m_{\mathrm{p}}} \Rightarrow \Delta v_j = \frac{\frac{\phi M_0}{N} v_{\mathrm{rel}}}{\left(M_0 - j\frac{\phi M_0}{N}\right) + \frac{\phi M_0}{N}} = v_{\mathrm{rel}} \frac{\phi}{\left(N - j\phi + \phi\right)} \quad . \tag{S1}$$

Here, as in section 3 of the main article, we calculate the impulse Δv_j due to expelling the j'th pellet of mass $m_{\mathrm{p}} = \phi M_0 / N$ when the rocket's remaining mass is $m = M_0 - j\phi M_0 / N$. The total Δv for constant relative velocity is therefore the sum of terms in equation (S1) from $j = 1$ to N. Table S1 lists the total Δv for all three of these discrete models, and for continuous thrust, in descending order of magnitude.

Comparison of the expressions for Δv **i**n Table S1 explains the different conclusions of the previous works cited in the main article. For a finite number of pellets, the "constant relative velocity" model predicts that for any fuel fraction ϕ, a single pellet or small number of pellets will produce a lower final Δv than if the fuel mass is expelled continuously. This stands in contrast to the other discrete propulsion models, which favor a single pulse ($N = 1$) over continuous thrust for maximising Δv.

These expressions are plotted in Figure S1, normalised such that their exhaust speeds agree in the limit of small ϕ (where the rocket's recoil from the exhaust can be neglected). Despite their different starting assumptions, for $N \to \infty$ all three discrete propulsion models converge to the continuous thrust rocket equation. The constant-energy-per-unit-mass model discussed in the main article (blue curves) lies between the two other discrete thrust models, starts closest (for N =1 and 2) to the continuous thrust rocket equation's Δv, and converges fastest towards that function as $N \to \infty$.

* In fact Tsiolkovsky wrote, *"...a constant relative velocity of the particles expelled must be taken as the foundation of rocket theory.*" See Reference [1] in the main article.

Table S1. Total Δv for three discrete propulsion models and for continuous thrust, in descending order of magnitude

Model assumptions	**Total Δv ($\phi = M_{fuel} / M_0$)**
N discrete pulses, constant impulse per unit mass v_p, from equation (10) with $\sqrt{2\alpha} = v_p$	$v_p \sum_{j=1}^{j=N} \frac{\phi}{N - j\phi}$
N discrete pulses, constant mechanical energy per unit mass α, from equation (8)	$\sqrt{2\alpha} \sum_{j=1}^{j=N} \frac{\phi}{\sqrt{(N - j\phi)(N - j\phi + \phi)}}$
Continuous thrust, from equation (1) with $u = v_{rel}$ and $m = M_0 - M_{fuel} = M_0(1 - \phi)$	$v_{rel} \ln\left(\frac{1}{1-\phi}\right)$
N discrete pulses, constant relative velocity v_{rel}, from equation (S1)	$v_{rel} \sum_{j=1}^{j=N} \frac{\phi}{(N - j\phi + \phi)}$

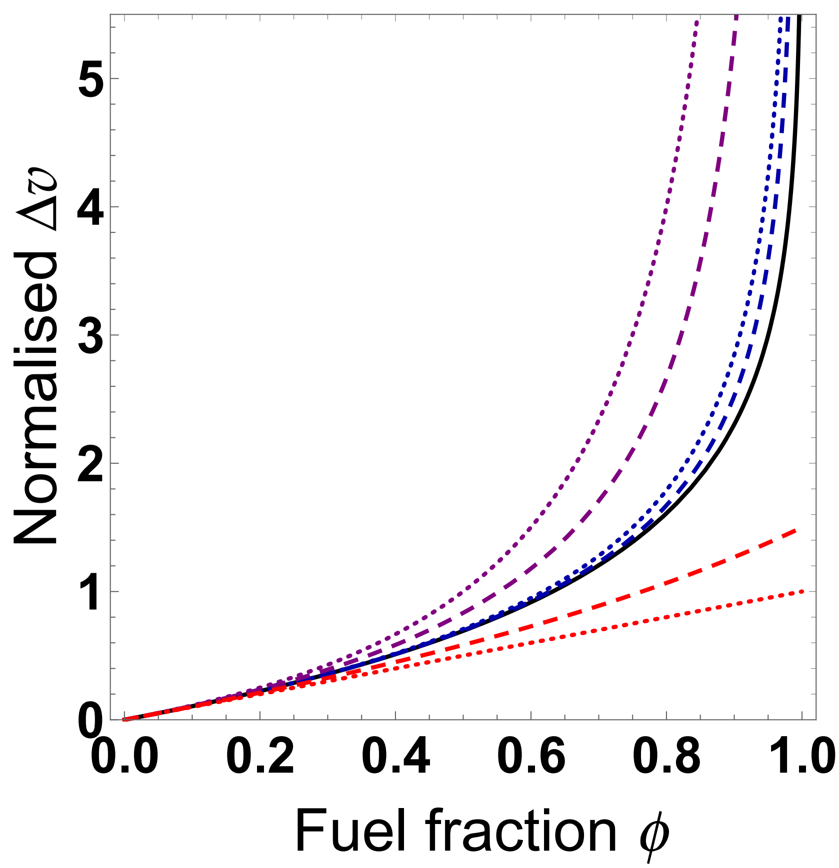

Figure S1. Comparison of normalised Δv for the continuous thrust model (black solid line) and three discrete propulsion models as described in the text and Table 1, for which the dotted and dashed curves correspond to $N = 1$ and $N = 2$, respectively. The blue curves are for a constant energy per unit mass, identical to those shown in figure (2) in the main article. The purple curves above them are for a constant impulse per unit mass, and the red curves below them are for a constant relative exhaust velocity (equation S1).