3.1 Basic Identities

<u>Remark</u>: Equations come in three flavors

1. Inconsistent equation (or impossible equation)

   $$x+2=x \qquad \text{Solution = empty set}$$

2. Conditional equation

   $$x+2=11 \qquad \text{Solution set} = \{9\}$$

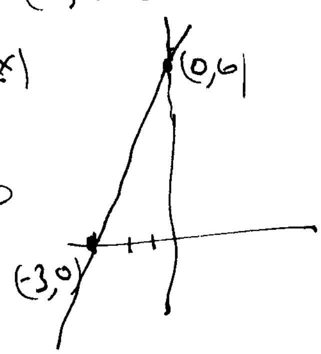
3. Identity

   $$2(x+3)=2x+6 \qquad \text{Solution = all real numbers}$$

<u>Another way</u> to look at identities:

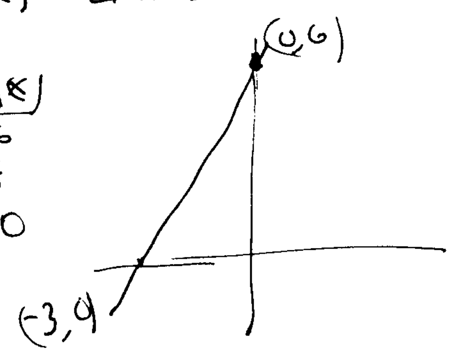
$f(x) = 2(x+3)$

x	$f(x)$
0	6
1	8
2	10



$g(x) = 2x+6$

x	$g(x)$
0	6
1	8
2	10



The two functions, f and g, are really the same function.

Pythagorean identities

$\sin^2 x + \cos^2 x = \underline{1}$

$\tan^2 x + 1 = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

Reciprocal Identities

$\csc x = \dfrac{1}{\sin x}$ $\qquad \sec x = \dfrac{1}{\cos x}$ $\qquad \cot x = \dfrac{1}{\tan x}$

Quotient Identities

$\tan x = \dfrac{\sin x}{\cos x}$ $\qquad \cot x = \dfrac{\cos x}{\tan x}$

Odd and Even Identities

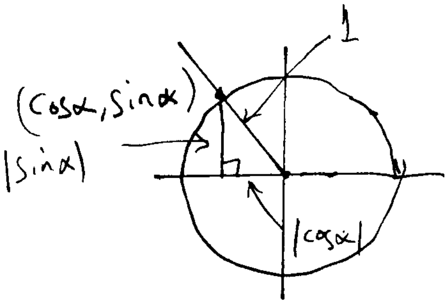
$\sin(-x) = -\sin x$ sine is a <u>odd</u> function

$\cos(-x) = \cos x$ cosine is an <u>even</u> function

$\tan(-x) = -\tan x$ tangent is an <u>odd</u> function

Why are these true? [How to derive these.]

why is $\sin^2\alpha + \cos^2\alpha = 1$ true?



Reciprocal identities? True by definition

Quotient identities?

$\sin\alpha = \frac{y}{r}$

$\cos\alpha = \frac{x}{r}$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{y}{\cancel{r}}}{\frac{x}{\cancel{r}}} \cdot \frac{\cancel{r}}{\cancel{r}} = \frac{y}{x} = \tan\alpha$$

Likewise $\frac{\cos\alpha}{\sin\alpha} = \frac{x/r}{y/r} = \frac{x}{y} = \cot\alpha$

Other pythagorean identities?

$\sin^2\alpha + \cos^2\alpha = 1$ Divide by $\cos^2\alpha$:

$$\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$$

$$\left(\frac{\sin\alpha}{\cos\alpha}\right)^2 + 1 = \left(\frac{1}{\cos\alpha}\right)^2$$

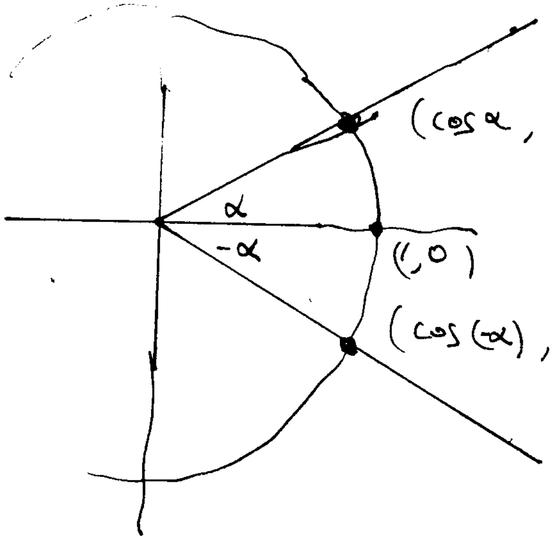
$$\tan^2\alpha + 1 = \sec^2\alpha$$

Likewise: $\frac{\sin^2\alpha}{\sin^2\alpha} + \frac{\cos^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha}$

$$1 + \left(\frac{\cos\alpha}{\sin\alpha}\right)^2 = \left(\frac{1}{\sin\alpha}\right)^2$$

$$1 + \cot^2\alpha = \csc^2\alpha$$

Even/odd?



Same x-coordinate

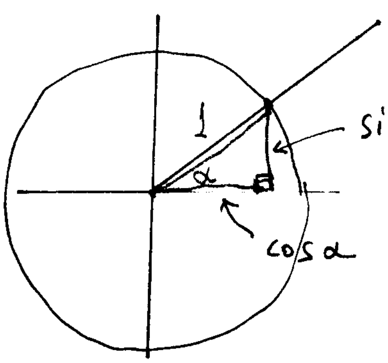
Opposite y-coordinate

So $\cos(-\alpha) = \cos\alpha$

$\sin(-\alpha) = -\sin\alpha$

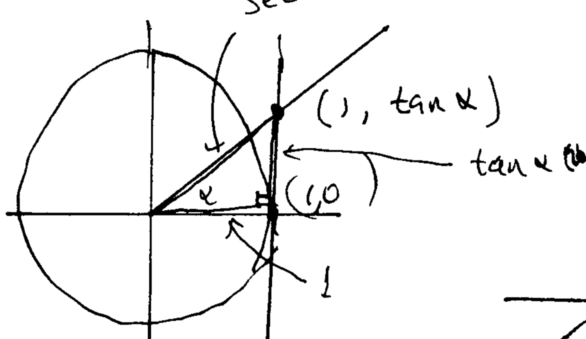
$$\tan(-\alpha) = \frac{\sin(-\alpha)}{\cos(-\alpha)} = \frac{-\sin\alpha}{\cos\alpha} = -\tan\alpha$$

A way to visualize the Pythagorean identities

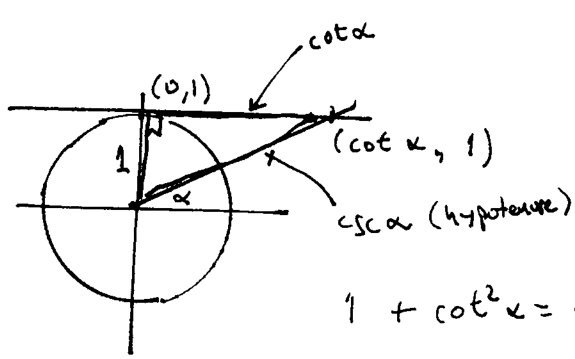


$\sin^2\alpha + \cos^2\alpha = 1$ ← hypotenuse

$\sec\alpha$ ← hypotenuse



$\tan^2\alpha + 1 = \sec^2\alpha$



$1 + \cot^2\alpha = \csc^2\alpha$

Remark: There is a lot of redundancy in the six trig functions.

ex: Given that α in QI and $\tan\alpha = \boxed{\frac{1}{2}}$. Find the value of the other five.

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{2}{1} = \boxed{2}$$

Use $\tan^2\alpha + 1 = \sec^2\alpha$

$$\sec^2\alpha = \left(\frac{1}{2}\right)^2 + 1 = \frac{1}{4} + \frac{4}{4} = \frac{5}{4}$$

$$\sec\alpha = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{\sqrt{4}} = \pm\frac{\sqrt{5}}{2} = \begin{cases} \boxed{\frac{\sqrt{5}}{2}} \leftarrow \text{QI} \\ \text{or} \\ \cancel{-\frac{\sqrt{5}}{2}} \end{cases}$$

$$\cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{\sqrt{5}/2} = \frac{2}{\sqrt{5}} = \boxed{\frac{2\sqrt{5}}{5}}$$

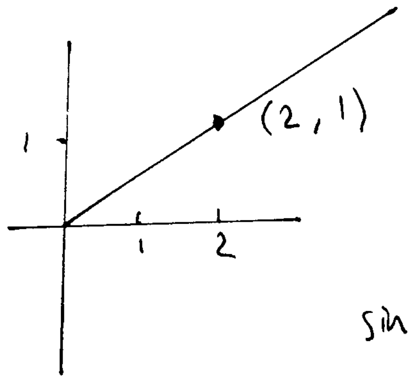
Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ so $\frac{1}{2} = \frac{\sin\alpha}{2/\sqrt{5}}$

$$\sin\alpha = \frac{2}{\sqrt{5}} \cdot \frac{1}{2} = \frac{1}{\sqrt{5}} = \boxed{\frac{\sqrt{5}}{5}}$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{1/\sqrt{5}} = \boxed{\sqrt{5}}$$

Now let's do the same problem geometrically.

ex Given that $\tan\alpha = \frac{1}{2}$ and α in QI find the other fns..



Take $x = 2$, $y = 1$

so $r = \sqrt{2^2+1^2} = \sqrt{5}$

$\sin\alpha = \frac{1}{\sqrt{5}}$ $\qquad \csc\alpha = \frac{\sqrt{5}}{1}$

$\cos\alpha = \frac{2}{\sqrt{5}}$ $\qquad \sec\alpha = \frac{\sqrt{5}}{2}$

$\tan\alpha = \frac{1}{2}$ $\qquad \cot\alpha = \frac{2}{1}$

ex. Given that $\cos\alpha = \cos\alpha$ find the other five trig functions; given that α is in QI.

That is, express the other trig functions in terms of $\cos\alpha$.

$\cos\alpha = \boxed{\cos\alpha}$

$\sec\alpha = \boxed{\frac{1}{\cos\alpha}}$

Use $\sin^2\alpha + \cos^2\alpha = 1$ to solve for $\sin\alpha$:

$\sin^2\alpha = 1 - \cos^2\alpha$

$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \begin{cases} \boxed{\sqrt{1-\cos^2\alpha}} & \leftarrow \alpha \text{ is in QI} \\ \text{or} \\ \cancel{-\sqrt{1-\cos^2\alpha}} \end{cases}$

$\csc\alpha = \frac{1}{\sin\alpha} = \boxed{\frac{1}{\sqrt{1-\cos^2\alpha}}}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \boxed{\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}}$

$\cot\alpha = \frac{1}{\tan\alpha} = \boxed{\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}}$

<u>Simplify</u>

33) $\sin(y) + \sin(-y) = \sin y + [-\sin y]$

$= \sin y - \sin y = \boxed{0}$

38) $(1 - \cos(-\alpha))(1 + \cos(\alpha))$

$= (1 - \cos\alpha)(1 + \cos\alpha)$

$= 1^2 - (\cos\alpha)^2 = 1 - \cos^2\alpha = \boxed{\sin^2\alpha}$

because $\sin^2\alpha + \cos^2\alpha = 1$

so $\sin^2\alpha = 1 - \cos^2\alpha$

or $\cos^2\alpha = 1 - \sin^2\alpha$