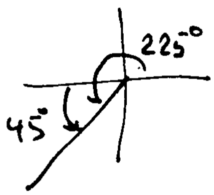
Warmup

(1) $\cos 225^\circ = ?$



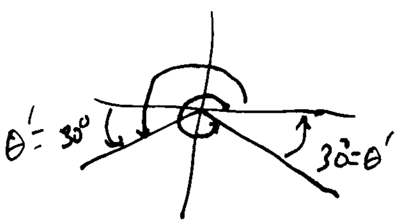
Reference angle $= 45^\circ$

$\cos 45^\circ = \frac{\sqrt{2}}{2}$

$\cos 225^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$

(2) $\sin\theta = -\frac{1}{2}$ Look for θ' so that $\sin\theta' = \frac{1}{2}$

Like $\theta' = 30^\circ$



$\sin\theta$ should be negative in Q III or Q IV

In Q IV take $\theta = 330^\circ$

In Q III take $\theta = 210^\circ$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ where θ is in radians?

Take $\theta = -\frac{\pi}{6} = -30^\circ$

3.4 Sum and Difference for sine and tangent

$$\boxed{\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}}$$

ex. $\sin 75^\circ = \sin(30^\circ + 45^\circ)$

$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$

ex. In the formula for $\sin(\alpha-\beta)$ what $\alpha=\beta$?

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad \text{Set } \alpha=\beta$$

$$0 = \sin(0) = \sin(\alpha-\alpha) = \sin\alpha\cos\alpha - \cos\alpha\sin\alpha$$
$$= \sin\alpha\cos\alpha - \sin\alpha\cos\alpha = 0$$

proof of the sum formula, based on the diff. formula for cosine.

$$\sin(\alpha+\beta) \underset{\text{cofunction identity}}{=} \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) \overset{\text{algebra}}{=} \cos\left(\frac{\pi}{2}-\alpha-\beta\right)$$

$$= \cos\left(\left(\frac{\pi}{2}-\alpha\right)-\beta\right)$$

diff. for cosine

$$= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$

$$\underset{\text{cofunction ID}}{=} \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \text{and we're done!}$$

[See the textbook for $\sin(\alpha-\beta)$ ...]

$$\boxed{\begin{aligned} \tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad \text{tangent of a sum} \\ \tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \quad \text{tangent of a difference} \end{aligned}}$$

ex: $\tan 15^\circ = \tan(45^\circ-30^\circ) = \dfrac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \cdot \tan 30^\circ}$

$$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} \cdot \frac{3}{3} = \boxed{\frac{3-\sqrt{3}}{3+\sqrt{3}}} = \frac{(3-\sqrt{3})}{(3+\sqrt{3})}\cdot\frac{(3-\sqrt{3})}{(3-\sqrt{3})}$$

$$= \frac{9-6\sqrt{3}+3}{9-3} = \frac{12-6\sqrt{3}}{6} = \boxed{2-\sqrt{3}}$$

proof of sum for tangent: "parents" of this ID: (1) $\sin(\alpha+\beta)$
(2) $\cos(\alpha+\beta)$

sum for sin and cos

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \cdot \frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cdot\cos\beta}}$$

$$= \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha}\cancel{\cos\beta}}{\cancel{\cos\alpha}\cancel{\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$ and we're done!

Quotient ID

[For $\tan(\alpha-\beta)$ see the textbook.]

26) Simplify $$\frac{\tan\left(\frac{\pi}{3}\right) - \tan\frac{\pi}{5}}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{5}\right)} = \tan\left(\frac{\pi}{3} - \frac{\pi}{5}\right)$$
$$= \tan\left(\frac{5\pi}{15} - \frac{3\pi}{15}\right) = \tan\left(\frac{2\pi}{15}\right)$$

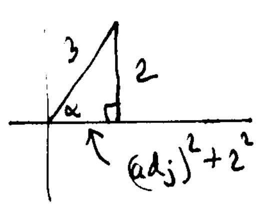
35) Find the exact value of $\sin(\alpha+\beta)$ if $\sin\alpha = \frac{2}{3}$ and $\sin\beta = -\frac{1}{2}$, where α is in Q I and β is in Q III.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \left(\frac{2}{3}\right)\cos\beta + \cos\alpha\left(-\frac{1}{2}\right)$$

Need these.

Find $\cos\alpha$ using geometry: Since α is in Q I, α is an acute angle.

$\sin\alpha = \frac{opp}{hyp} = \frac{2}{3}$

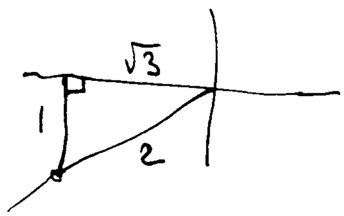


$(adj)^2 + 2^2 = 3^2 \Rightarrow adj = \sqrt{3^2-2^2} = \sqrt{9-4} = \sqrt{5}$

$\cos\alpha = \frac{adj}{hyp} = \boxed{\frac{\sqrt{5}}{3}}$

Find $\cos\beta$ using geometry again: $\sin\beta = -\frac{1}{2}$ and β is in Q III

So $\sin\beta' = \frac{1}{2} = \frac{opp}{hyp}$



$\cos\beta' = \frac{adj}{hyp} = \frac{\sqrt{3}}{2}$ but cos in neg in Q III

So $\cos\beta = \boxed{-\frac{\sqrt{3}}{2}}$

[To find $\cos\beta$ using a pythagorean identity:

Use $\sin^2\beta + \cos^2\beta = 1$

$\left(-\frac{1}{2}\right)^2 + \cos^2\beta = 1 \Rightarrow \frac{1}{4} + \cos^2\beta = \frac{4}{4}$

$\cos^2\beta = \frac{3}{4}$

choose '−'

$\cos\beta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2} = \boxed{-\frac{\sqrt{3}}{2}}$]

To finish the problem:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \left(\frac{2}{3}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{5}}{3}\right)\left(-\frac{1}{2}\right)$$
$$= \frac{-2\sqrt{3}}{6} - \frac{\sqrt{5}}{6} = \boxed{-\frac{(2\sqrt{3}+\sqrt{5})}{6}}$$

3.5 Double angle

$$\boxed{\begin{aligned} \sin 2x &= 2\sin x\cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x} \end{aligned}}$$

Reason: Take $\alpha = x$, $\beta = x$

$\sin(x+x) = \sin x\cos x + \cos x\sin x$

$\sin(2x) = 2\sin x\cos x$