Loose ends in 3.5 Half-Angle Identity

10) Find $\cos 15^\circ$. Use $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

$$\cos 15^\circ = \cos\frac{30^\circ}{2} = +\sqrt{\frac{1+\cos 30^\circ}{2}}$$
$$= \sqrt{\frac{1}{2}\left[1+\frac{\sqrt{3}}{2}\right]} = \sqrt{\frac{1}{2}\left[\frac{2+\sqrt{3}}{2}\right]}$$
$$= \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{\sqrt{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

DECLARATION: The half-angle identities are the last you must memorize.

3.6 Product-to-Sum Identities

(1) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(2) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\cos(A-B)+\cos(A+B) = 2\cos A\cos B \Rightarrow$$
$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Similarly: Subtract (2) − (1)

$$\cos(A-B)-\cos(A+B) = 2\sin A\sin B \Rightarrow$$
$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$

ex: 8) Suppose $f(t) = \cos 3t$ and $g(t) = \sin 5t$. Find a way to rewrite $f(t)\cdot g(t) = \cos 3t\sin 5t$.

Use $\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

with $A = 3t$ and $B = 5t$.

8 cont'd) $\cos 3t \sin 5t = \frac{1}{2}\left[\sin(3t+5t) - \sin(3t-5t)\right]$

$$= \frac{1}{2}\left[\sin 8t - \sin(-2t)\right]$$

$$= \frac{1}{2}\left[\sin 8t + \sin 2t\right]$$

Sum-to-Product identities

We know $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

That is $\cos(A-B) + \cos(A+B) = 2\cos A \cos B$

Shoot! I wish that $A+B = x$, and $A - B = y$. Then everything would be good.

We can arrange this:

Add: $2A = x+y$ so $A = \frac{x+y}{2}$

Subtract: $2B = x-y$ so $B = \frac{x-y}{2}$.

Substitute:

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

[The other three can be derived similarly]

Use this with $x = \frac{2}{3}, y = \frac{1}{2}$

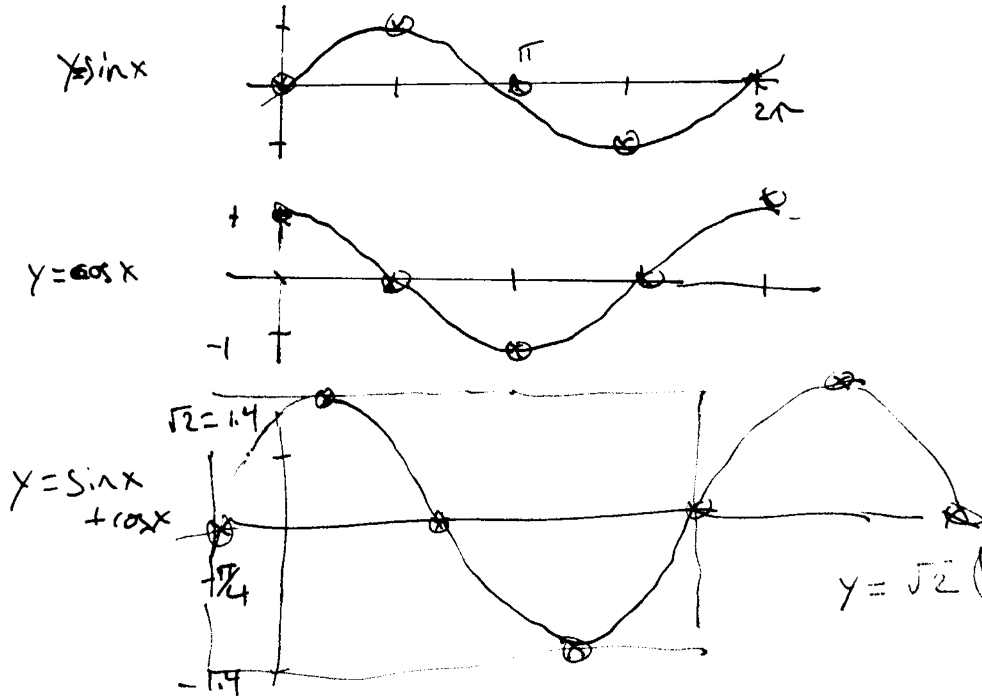
26) Rewrite $\cos\left(\frac{1}{2}\right) + \cos\left(\frac{2}{3}\right)$

$$= 2\cos\left[\frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right)\right]\cos\left[\frac{1}{2}\left(\frac{2}{3}-\frac{1}{2}\right)\right]$$

$$= 2\cos\left[\frac{1}{2}\left(\frac{7}{6}\right)\right]\cos\left[\frac{1}{2}\left(\frac{1}{6}\right)\right]$$

$$= 2\cos\left(\frac{7}{12}\right)\cos\left(\frac{1}{12}\right)$$

ex: What is your prediction of what this graph of $y = \sin x + \cos x$ looks like?



$y = \sqrt{2}\left[\sin\left(x + \frac{\pi}{4}\right)\right]$

what's happening?

$$y = \sin x + \cos x = 1 \cdot \sin x + 1 \cdot \cos x$$

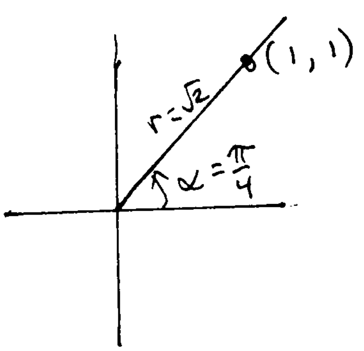
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cdot \cos x\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{2}}{2} \sin x + \frac{\sqrt{2}}{2} \cos x\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} \cdot \sin x + \sin\frac{\pi}{4} \cdot \cos x\right)$$

$$= \sqrt{2}\left(\sin x \cdot \cos\frac{\pi}{4} + \cos x \cdot \sin\frac{\pi}{4}\right)$$

$$y = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)$$



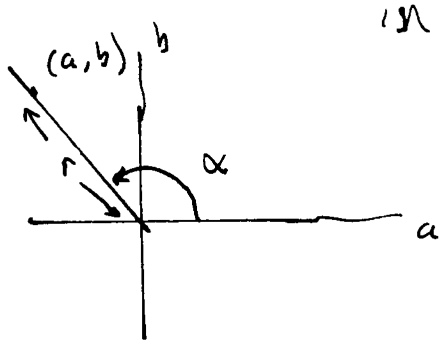
So the amplitude is $\sqrt{2}$
and the phase shift is $-\frac{\pi}{4}$

Reduction Formula:

$a \sin x + b \cos y$ can be written as

$$\boxed{r \sin(x+\alpha)}$$

where $r = \sqrt{a^2+b^2}$ and α is the angle in this picture:

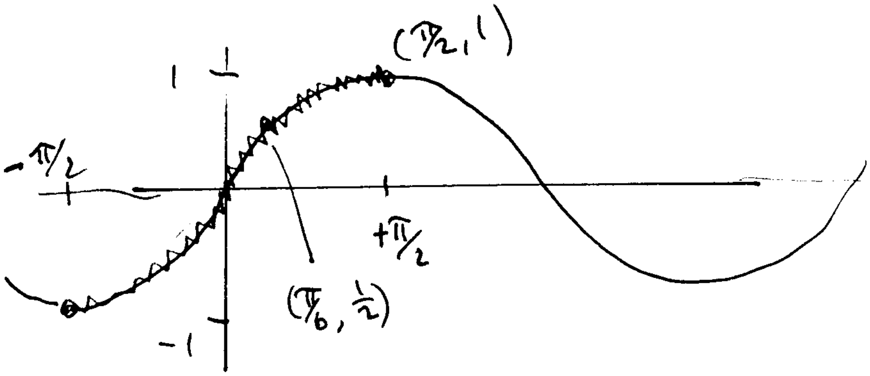


so that $\sin\alpha = \frac{b}{r} = \frac{b}{\sqrt{a^2+b^2}}$

and $\cos\alpha = \frac{a}{r} = \frac{a}{\sqrt{a^2+b^2}}$

Recall: Five ways to view functions
1. Machine
2. Equation
3. Table
4. Graph
5. Mapping.

4.1 Inverse Trig Functions

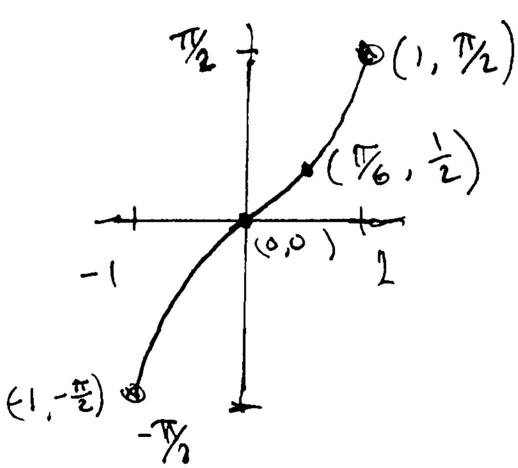


Restrict the domain of $y = \sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
so the new function is one-to-one.

Call this restricted function

$y = \text{Sin}\, x$ Domain $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range $= [-1, 1]$



Since Sine is one-to-one, its inverse function exists.
Call it $y = \arcsin(x)$.

Domain of arcsine $= [-1, 1]$

Range of arcsine $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Defn: $y = \arcsin x$ means

(i) $\sin y = x$

and (2) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

example Find $\arcsin\left(\frac{\sqrt{2}}{2}\right)$.

$y = \arcsin \frac{\sqrt{2}}{2}$ says (i) $\sin y = \frac{\sqrt{2}}{2}$

and (2) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

so $y = \frac{\pi}{4}$.