5.5 Double Angle Identities

<u>Know</u>: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

what if A and B are equal? Set $A=B$.

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$$

Double angle formula for cosine (1st form)

<u>Use</u>: $\sin^2 A = 1 - \cos^2 A$ ← pythagorean id.

$$\begin{aligned}\cos 2A &= \cos^2 A - (1 - \cos^2 A)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

$$\boxed{\cos 2A = 2\cos^2 A - 1}$$

(2nd form)

<u>Use</u>: $\cos^2 A = 1 - \sin^2 A$ to sub into 1st form

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A$$

$$\boxed{\cos 2A = 1 - 2\sin^2 A}$$

(3rd form)

<u>Know</u>: $\sin(A+B) = \sin A \cos B + \cos A \sin B$, and if $A=B$,

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$

$$\boxed{\sin 2A = 2 \sin A \cos A}$$

Double angle formula for sine

<u>Know</u>. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, and if $A=B$

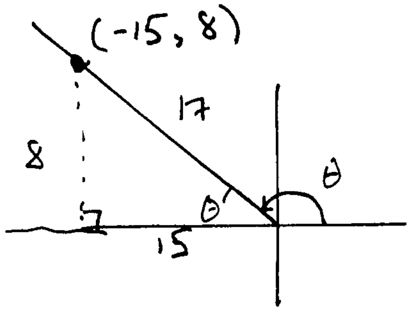
$$\boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Double angle formula for tangent

ex: Given $\sin\theta = \frac{8}{17}$ and $\cos\theta < 0$.

a) Find $\sin 2\theta$ b) Find $\cos 2\theta$ c) Find $\tan 2\theta$.

a) $\sin 2\theta = 2\underbrace{\sin\theta}_{\text{given}}\underbrace{\cos\theta}_{\text{need}}$



← Shows $\cos\theta' = \frac{15}{17}$ and

$\cos\theta = -\frac{15}{17}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{8}{17}\right)\left(\frac{-15}{17}\right) = \boxed{-\frac{240}{289}}$$

b) $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2(\sin\theta)^2$

$$= 1 - 2\left(\frac{8}{17}\right)^2 = \boxed{\frac{161}{289}}$$

Note: we never used $\cos\theta < 0$.

c) $$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-240/289}{161/289} = \boxed{-\frac{240}{161}}$$

ex: a) Simplify the expression $2\cos^2 5x - 1$

Use: $2\cos^2 A - 1 = \cos 2A$, with $A = 5x$.

$$2\cos^2 5x - 1 = \cos 2(5x) = \boxed{\cos 10x}$$

b) Simplify the expression $\sin 165^\circ \cos 165^\circ$.

Use: $2\sin A\cos A = \sin 2A$

$$\sin 165^\circ\cos 165^\circ = \frac{1}{2}(2\sin 165^\circ\cos 165^\circ) = \frac{1}{2}(\sin 2\cdot 165^\circ)$$

$$= \frac{1}{2}\sin 330^\circ = \frac{1}{2}\left(-\frac{1}{2}\right)$$

$$= \boxed{-\frac{1}{4}}$$

ex: Write $\cos 3x$ in terms of $\cos x$.

$$\cos 3x = \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x$$ sum formula for cosine

$$= (\cos^2 x - \sin^2 x)\cos x - 2\sin x \cos x \cdot \sin x$$ Double angle IDs

$$= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x$$

$$= \cos^3 x - 3\sin^2 x \cos x$$ Pythagorean ID

$$= \cos^3 x - 3\cos x(1 - \cos^2 x)$$

$$= \cos^3 x - 3\cos x + 3\cos^3 x$$

$$= 4\cos^3 x - 3\cos x$$

Remark: In retrospect, using the 2nd form of double-angle ID for cosine would have saved a step or two.

Product-to-Sum identities Here is a trick for deriving one of the identities:

Know: (1) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(2) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(1) + (2): $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$ multiply by $\frac{1}{2}$:

$$\boxed{\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]}$$

ex: $\cos 8x \cos 3x = \frac{1}{2}\left[\cos(8x+3x) + \cos(8x-3x)\right]$

$$= \frac{1}{2}\left[\cos 11x + \cos 5x\right]$$