5.6 Half-Angle Identities

Recall: $\cos 2x = 2\cos^2 x - 1$ — use algebra to solve for $\cos^2 x$:

Add 1: $1 + \cos 2x = 2\cos^2 x$

Divide by 2: $\dfrac{1+\cos 2x}{2} = \cos^2 x$

Remark: If we stop here, this a "power reducing identity"

Take square roots: $\cos x = \pm\sqrt{\dfrac{1+\cos 2x}{2}}$

Let $2x = A$, So $x = \dfrac{A}{2}$.

$$\boxed{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

Half angle ID for cosine

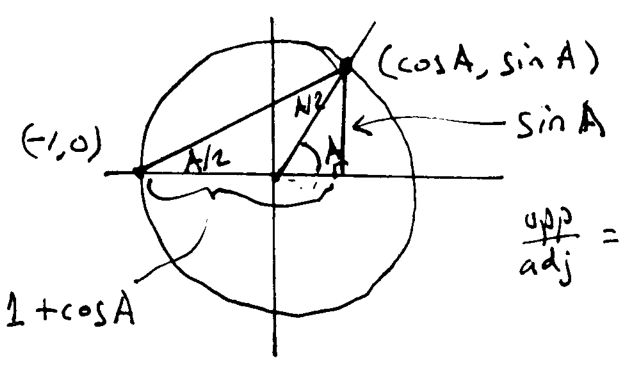
Remark: Had we started with $\cos 2x = 1 - 2\sin^2 x$ we would similarly derive

$$\boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Half-angle ID for Sine

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\frac{\sqrt{\frac{1-\cos A}{2}}}{\sqrt{\frac{1+\cos A}{2}}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\boxed{\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}} = \frac{1-\cos A}{\sin A}$$



$\dfrac{\text{opp}}{\text{adj}} = \tan\dfrac{A}{2} = \dfrac{\sin A}{1+\cos A}$

ex: Find the exact value of $\sin 22.5°$.

Note: $22.5° = \frac{45°}{2}$ Use $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

with $A = 45°$.

$$\sin 22.5° = \sin\frac{45°}{2} = \pm\sqrt{\frac{1-\cos 45°}{2}} =$$

Choose '+'
because sine
is positive in Q I

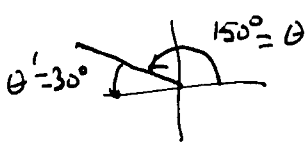
$$\hookrightarrow = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2}{2}\cdot\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2-\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}}{2\cdot 2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

ex: Find the exact value of $\tan 75°$ by using

$$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}, \quad \text{with } A = 150°$$

$$\tan 75° = \tan\frac{150°}{2} = \frac{\sin 150°}{1+\cos 150°} = \frac{\frac{1}{2}}{1+\left(-\frac{\sqrt{3}}{2}\right)}$$



$$= \frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}\cdot\frac{2}{2} = \frac{1}{2-\sqrt{3}} =$$

optional: Rationalize the denominator

$$\hookrightarrow = \frac{1}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{2^2-\sqrt{3}^2} = \frac{2+\sqrt{3}}{4-3}$$

$$= \frac{2+\sqrt{3}}{1} = 2+\sqrt{3}$$

ex: Simplify $\pm\sqrt{\frac{1-\cos 8x}{2}}$

Note: This looks like $\pm\sqrt{\frac{1-\cos A}{2}}$ with $A = 8x$

So this should equal $\sin\frac{A}{2} = \sin\frac{8x}{2} = \boxed{\sin 4x}$

(where '+' or '−' are chosen depending on which quadrant $4x$ is in.)

5.6 ~~20~~ Find $\sin\frac{x}{2}$ given that $\cos x = \frac{5}{8}$ with $\frac{\pi}{2} < x < \pi$.

that is, x in quadrant II.

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

NOTE: $\frac{1}{2}\cdot\frac{\pi}{2} < \frac{1}{2}\cdot x < \frac{1}{2}\cdot\pi$

so $\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

That is, $\frac{x}{2}$ is in QI.

Choose '+'.

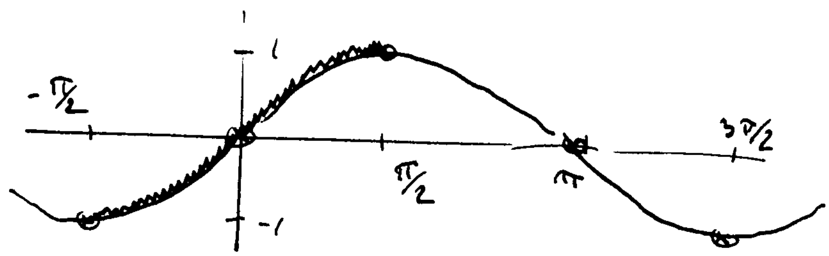
$$\sin\frac{x}{2} = \sqrt{\frac{1-\frac{5}{8}}{2}}$$

$$= \sqrt{\frac{1}{2}\left(1-\frac{5}{8}\right)} = \sqrt{\frac{1}{2}\left(\frac{8}{8}-\frac{5}{8}\right)}$$

$$= \sqrt{\frac{1}{2}\left(\frac{3}{8}\right)} = \sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{\sqrt{16}} = \frac{\sqrt{3}}{4}$$

6.1 Inverse Circular Functions

We want to define the "inverse sine function".



x	$\sin x$
$-\pi/2$	-1
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

<u>Note</u>: The graph $y = \sin x$ <u>fails</u> the horizontal line test.

Define a <u>new</u> function by restricting the domain of sine just enough to make it one-to-one, namely restrict x by: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

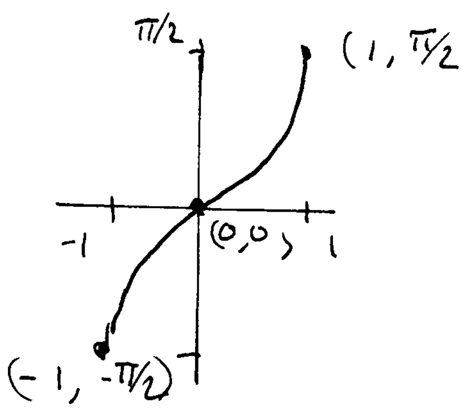
Remark: Some books call this restricted sine function

$\operatorname{Sin} x$, to indicate its a different function.

Call the inverse function of this one-to-one function:

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$

x	$\arcsin x$
-1	$-\pi/2$
0	0
$1/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$\sqrt{3}/2$	$\pi/3$
1	$\pi/2$



Domain of arcsin $= [-1, 1]$

Range of arcsin $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$