16.7 Surface Integrals

<u>Remark</u>: Just like with line integrals, there are two types of surface integrals

(1) $\iint_S f(x,y,z)\,dS$, a surface integral of the real-valued function $f(x,y,z)$; dS = element of surface area

Physical intuition: $f(x,y,z)$ = mass density (g/cm^2) of a surface, and

$\iint_S f\,dS$ = total mass

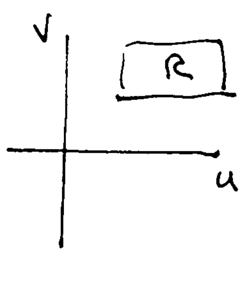
(2) $\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_S \vec{F}(x,y,z)\cdot\vec{n}\,dS$, "flux".

Physical intuition: $\vec{F}(x,y,z)$ = velocity of fluid,

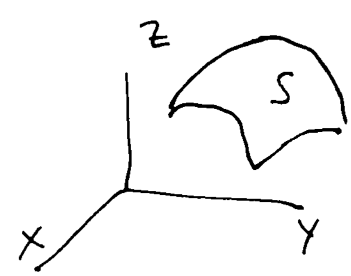
S = a membrane, $\iint_S \vec{F}\cdot d\vec{S}$ = amount of mass passing through the membrane.

<u>How to calculate</u> $\iint_S f(x,y,z)\,dS$

Given a parametrized surface



$$\longrightarrow \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$



<u>OR</u> $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$$\iint_S f(x,y,z)\,dS = \iint_R f(x(u,v), y(u,v), z(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

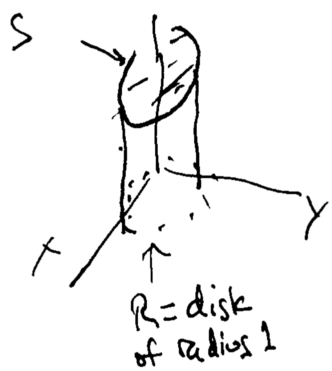
Remark: (1) In the special case that $f(x,y,z)=1$,
this calculated the area of S.
(2) If you're not given a parametrization for S,
you need to design your own parametrization.

ex: $f(x,y,z) = x^2+y^2+z^2$

$S: \; z = x+2, \quad x^2+y^2 \le 1$

plane

inside a cylinder of radius 1



we need to parameterize the plane.

$$\begin{cases} x(u,v) = u \\ y(u,v) = v \\ z(u,v) = u+2 \end{cases}$$

so $\vec{r}(u,v) = \langle u, v, u+2 \rangle$

$\vec{r}_u(u,v) = \langle 1, 0, 1 \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, 0 \rangle$

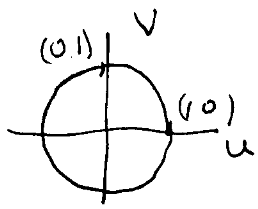
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \langle -1, 0, 1 \rangle$$

a normal vector to the plane S

$$dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$
$$= \sqrt{(-1)^2 + 0^2 + (1)^2} \, du \, dv = \sqrt{2} \, du \, dv$$

$$\iint_S f(x,y,z) \, dS = \iint_S (x^2+y^2+z^2) \, dS$$
$$= \iint_R \left(u^2 + v^2 + (u+2)^2\right) \sqrt{2} \, du \, dv$$

where R is:



$$= \iint_R \left(u^2+v^2+u^2+4u+4\right)\sqrt{2} \, du \, dv$$
$$= \iint_R \left(2u^2+4u+4+v^2\right)\sqrt{2} \, du \, dv$$

Now use polar coordinates:

$u = r\cos\theta$

$v = r\sin\theta$

$du\,dv = r\,dr\,d\theta$

$u^2+v^2 = r^2$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\left(2r^2\cos^2\theta + 4r\cos\theta + 4 + r^2\sin^2\theta\right) r \, dr \, d\theta$$
$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\left[2r^2\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right) + 4r\cos\theta + 4 + r^2\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)\right] r \, dr \, d\theta$$

[continued ...]

[This page a continuation of previous example, finished after class.]

$$\cdots = \sqrt{2}\int_0^{2\pi}\int_0^1 \left(r^2 + r^2\cos 2\theta + 4r\cos\theta + 4 + \tfrac{1}{2}r^2 - \tfrac{1}{2}r^2\cos 2\theta\right)\cdot r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \left(\tfrac{3}{2}r^3 + 4r + \tfrac{1}{2}r^3\cos 2\theta + 4r^2\cos\theta\right)dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\tfrac{3}{8}r^4 + 2r^2 + \tfrac{1}{8}r^4\cos 2\theta + \tfrac{4}{3}r^3\cos\theta\right]_0^1 d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\tfrac{3}{8} + 2 + \tfrac{1}{8}\cos 2\theta + \tfrac{4}{3}\cos\theta\right)d\theta$$

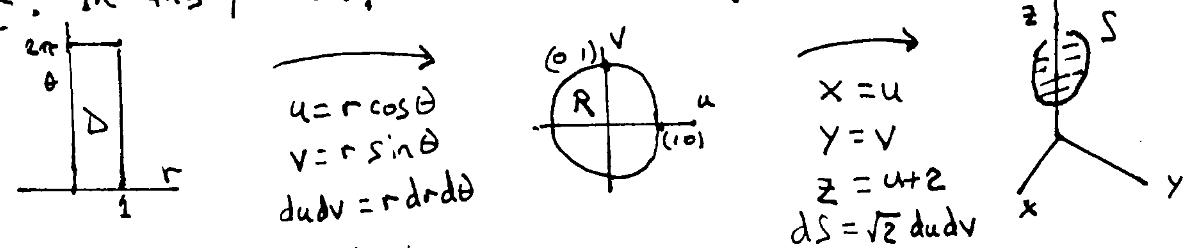
$$= \sqrt{2}\int_0^{2\pi}\left(\tfrac{19}{8} + \tfrac{1}{8}\cos 2\theta + \tfrac{4}{3}\cos\theta\right)d\theta$$

$$= \sqrt{2}\left[\tfrac{19}{8}\theta + \tfrac{1}{16}\sin 2\theta + \tfrac{4}{3}\sin\theta\right]_0^{2\pi}$$

$$= \sqrt{2}\left(\tfrac{19}{8}\cdot 2\pi + 0 + 0\right) = \boxed{\tfrac{19}{4}\sqrt{2}\,\pi}$$

That is, $\displaystyle\iint_S (x^2+y^2+z^2)\,dS = \frac{19\sqrt{2}\pi}{4}$

Remark: In this problem, we did two changes of variables:



We could have composed these into one parametrization:

$x = r\cos\theta$

$y = r\sin\theta$

$z = r\cos\theta + 2$

with $dS = \sqrt{2}\,du\,dv = \sqrt{2}\,r\,dr\,d\theta$

<u>Defn</u> : $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$

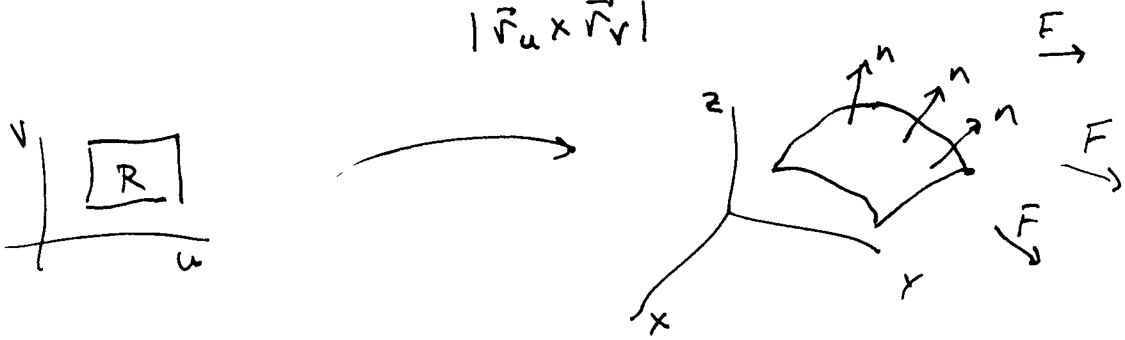
is the surface integral of a vector field $\vec{F}$ over the (oriented) surface S.

<u>How to calculate</u> this, given a parametrization

$\vec{r}(u,v)$ with domain R in the uv-plane

$\vec{r}_u(u,v) \times \vec{r}_v(u,v)$ = a normal vector to S
(but maybe not a <u>unit</u> normal!)

Let $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$. Then



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

$$= \iint_R \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \iint_R \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \iint_R F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

ex: $S: \; x^2+y^2+z^2 = 9$

with outward pointing unit normal $\vec{n}$

parametrization for S:

$x(u,v) = 3\sin u \cos v$

$y(u,v) = 3\sin u \sin v$

$z(u,v) = 3\cos u$

$0 \le u \le \pi, \qquad 0 \le v \le 2\pi$

NOTE: In spherical $u=\phi$, $v=\theta$, $\rho=3$ is the inspiration.

equivalent: $\vec{r}(u,v) = \langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u\rangle$

$$\vec{r}_u = \langle 3\cos u\cos v,\ 3\cos u\sin v,\ -3\sin u\rangle$$

$$\vec{r}_v = \langle -3\sin u\sin v,\ 3\sin u\cos v,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos u\cos v & 3\cos u\sin v & -3\sin u \\ -3\sin u\sin v & 3\sin u\cos v & 0 \end{vmatrix}$$

$$[\ldots] = 9\langle \sin^2 u\cos v,\ \sin^2 u\sin v,\ \sin u\cos u\rangle$$

$$= 9\sin u\langle \sin u\cos v,\ \sin u\sin v,\ \cos u\rangle$$

ex (cont'd) [Added after class.]

Find the flux: $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle 2x, 2y, 2z\rangle$
$= 2\langle x, y, z\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S 2\langle x,y,z\rangle\cdot\vec{n}\, dS$$

$$= \iint_R \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\, du\, dv$$

Since $F(x,y,z) = 2\langle x, y, z\rangle$,

$$\vec{F}(\vec{r}(u,v)) = 2\langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos v\rangle$$
$$= 6\langle \sin u\cos v,\ \sin u\sin v,\ \cos v\rangle.$$

And since $\vec{r}_u\times\vec{r}_v = 9\sin u\langle \sin u\cos v,\ \sin u\sin v,\ \cos v\rangle$,

$$\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v) = 6(9\sin u)\langle \sin u\cos v, \sin u\sin v, \cos v\rangle\cdot\langle \sin u\cos v, \sin u\sin v, \cos v\rangle$$
$$= 54\sin u\left[\sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 v\right]$$
$$= 54\sin u\left[\sin^2 u(\cos^2 v + \sin^2 v) + \cos^2 v\right] = 54\sin u\left[\sin^2 u + \cos^2 v\right]$$
$$= 54\sin u$$

$$\therefore \iint_R \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\, du\, dv = \int_0^{2\pi}\int_0^{\pi} 54\sin u\, du\, dv$$
$$= 54\int_0^{\pi}\sin u\, du\int_0^{2\pi} dv = 54\left[-\cos u\right]_0^{\pi}\left[v\right]_0^{2\pi}$$
$$= 54(2)(2\pi) = \boxed{216\pi}$$

[Added after class.]

Remarks on the previous example:

We can get some insight into the previous example by noting the sphere $S: x^2+y^2+z^2=9$ is a level surface of $g(x,y,z) = x^2+y^2+z^2$

So that, on S: $\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle = 2\langle x,y,z\rangle$

and $|\nabla g(x,y,z)| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{9} = 6$

hence $\left\{\begin{matrix}\text{outward}\\ \text{unit normal}\end{matrix}\right\} = \dfrac{\nabla g}{|\nabla g|} = \dfrac{2\langle x,y,z\rangle}{6} = \frac{1}{3}\langle x,y,z\rangle = \vec{n}$

Also, $\vec{F}(x,y,z) = 2\langle x,y,z\rangle$ so

$$\vec{F}\cdot\vec{n} = 2\langle x,y,z\rangle \cdot \frac{1}{3}\langle x,y,z\rangle$$

$$= \frac{2}{3}(x^2+y^2+z^2) = \frac{2}{3}(9) \quad \text{on } S, \text{ where } x^2+y^2+z^2=9.$$

$$= 6$$

Hence, $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S 6\,dS = 6\iint_S dS$

$= 6$ (Surface area of a sphere of radius 3)

$= 6\,(4\pi 3^2)$

$= \boxed{216\pi}$.