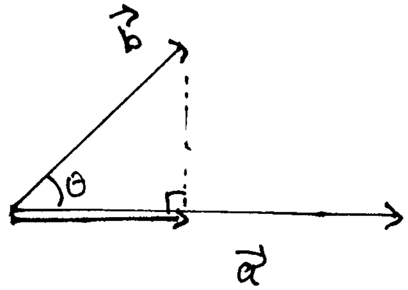
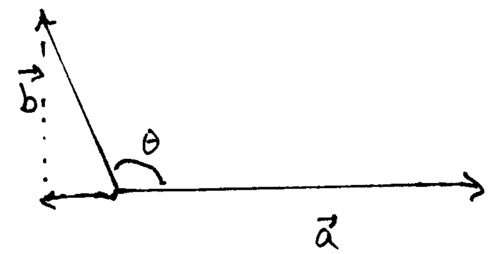
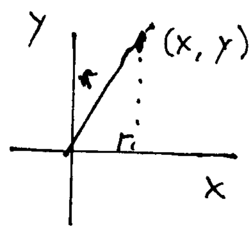
12.3 Dot Product (loose ends)

Scalar and vector projection





Recall trig



$\cos\theta = \frac{x}{r}$ so $x = r\cos\theta$

Let $r = |\vec{b}|$

So $x = r\cos\theta$

$$= |\vec{b}| \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

<u>Defn</u>: Scalar projection of $\vec{b}$ onto $\vec{a}$ $= \text{comp}_{\vec{a}}\, \vec{b} = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$

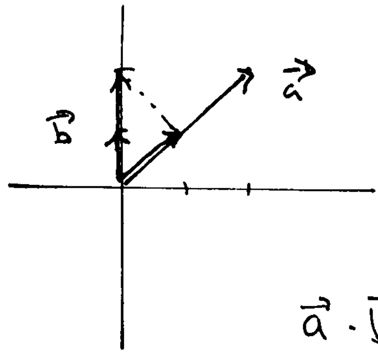
<u>Remark</u>: $\vec{a}\cdot\vec{b} = |\vec{a}| \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = |\vec{a}|\, \text{comp}_{\vec{a}}\, \vec{b}$

<u>Now</u>: unit in direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$ so the vector of (signed) length $\text{comp}_{\vec{a}}\, \vec{b}$ will be

$$(\text{comp}_{\vec{a}}\, \vec{b}) \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \frac{\vec{a}}{|\vec{a}|} = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\, \vec{a}} = \text{proj}_{\vec{a}}\, \vec{b}$$

Define to be the vector projection of $\vec{b}$ onto $\vec{a}$.

ex: $\vec{a} = \langle 2, 2 \rangle \qquad \vec{b} = \langle 0, 2 \rangle$



$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\vec{a} \cdot \vec{b} = \langle 2, 2 \rangle \cdot \langle 0, 2 \rangle = 4$$

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \frac{\vec{a}}{|a|} = \sqrt{2} \frac{\langle 2, 2 \rangle}{2\sqrt{2}}$$

$$= \frac{1}{2} \langle 2, 2 \rangle = \langle 1, 1 \rangle$$

12.4 Cross Product (only for vectors in $\mathbb{R}^3$)

Review of 2x2 and 3x3 determinants

$$\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = (2)(5) - (1)(3) = 10 - 3 = \boxed{7}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{array}{ccc} 12 & 12 & 5 \end{array}$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 5 & 4 & 2 \\ 1 & 3 & 1 \end{vmatrix} \begin{array}{cc} 2 & 1 \\ 5 & 4 \\ 1 & 3 \end{array} \qquad 8 + 2 + 45 - 12 - 12 - 5 = \boxed{26}$$

$$\begin{array}{ccc} 8 & 2 & 45 \end{array}$$

<u>Defn</u>: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

then the <u>cross product</u> $\vec{a} \times \vec{b}$ is defined as follows

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ a_1 & a_2 \\ b_1 & b_2 \end{matrix}$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

(equivalent to) $$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\vec{i} \times \vec{j} = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k} = \vec{k}$$

<u>Also</u> (check): $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

Fact $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Reason: If you swap two rows of a determinant, you toggle the sign.

Fact: $\vec{a} \times \vec{a} = \vec{0}$

Reason: $\vec{a} \times \vec{a} = -(\vec{a} \times \vec{a})$ so

$2(\vec{a} \times \vec{a}) = \vec{0}$ so $\vec{a} \times \vec{a} = \frac{1}{2}\vec{0} = \vec{0}$.

So $\vec{i} \times \vec{j} = -(\vec{j} \times \vec{i}) = \vec{k}$

$\vec{j} \times \vec{k} = -(\vec{k} \times \vec{j}) = \vec{i}$

$\vec{k} \times \vec{i} = -(\vec{i} \times \vec{k}) = \vec{j}$

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$

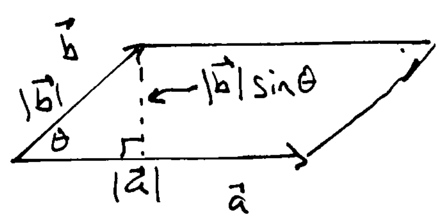
Important facts about cross products

(1) $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

[Defn: $\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$]

(2) (a) $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$, where θ = angle between a and b.

That is, $|\vec{a} \times \vec{b}|$ is the same as the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$



area = (base)(ht)
$= |\vec{a}|\,|\vec{b}| \sin\theta$

(b) The direction of $\vec{a} \times \vec{b}$ follows from (1) above and the "right hand rule".

12.4 1) $\vec{a} = \langle 6, 0, -2\rangle \qquad b = \langle 0, 8, 0\rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 8 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 6 & -2 \\ 0 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} k$$

$$= 16\vec{i} - 0\vec{j} + 48k = \langle 16, 0, 48\rangle$$

check: $\vec{a}\cdot(\vec{a}\times\vec{b}) = \langle 6, 0, -2\rangle\cdot\langle 16, 0, 48\rangle$

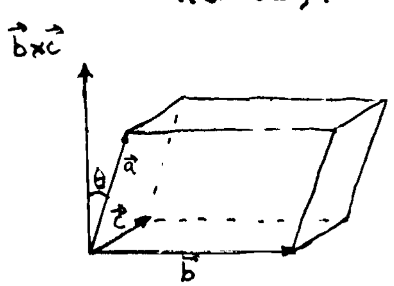
$= 96 + 0 - 96 = 0$

$\vec{b}\cdot(\vec{a}\times\vec{b}) = \langle 0, 8, 0\rangle\cdot\langle 16, 0, 48\rangle = 0+0+0 = 0$

Triple Product:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Remark: This "signed" volume is positive if $\theta < 90^\circ$ (so $\vec{a}, \vec{b}, \vec{c}$ follows the right hand rule) and is negative if $\theta > 90^\circ$ (so left-hand rule).



$= \pm$ volume of the parallelpiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$= \pm$ (height of parallelogram) (area of base of parallelogram)

$= (|\vec{a}|\cos\theta)\quad |\vec{b}\times\vec{c}|$

$= \left(|\vec{a}| \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{a}||\vec{b}\times\vec{c}|}\right) |\vec{b}\times\vec{c}| = \vec{a}\cdot(\vec{b}\times\vec{c})$