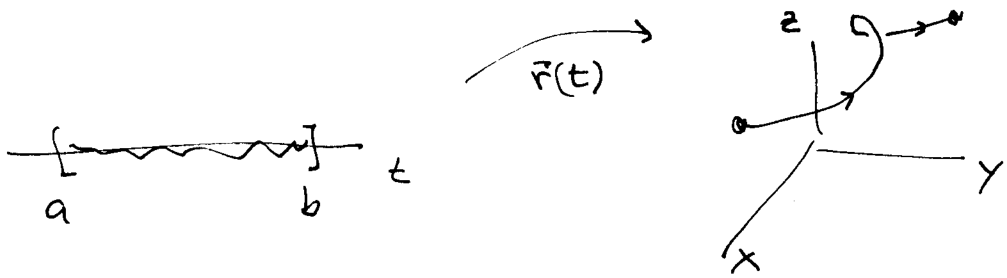
13.3 Arc length and (eventually) curvature

Remark: If $\vec{r}(t) = \text{position} = \langle x, y, z \rangle$

then $\vec{r}\,'(t) = \text{velocity} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

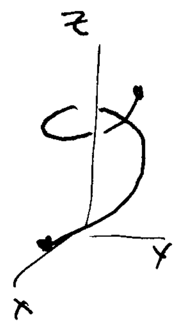
and $|\vec{r}\,'(t)| = \text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$



$$\text{arc length} = L = \int_a^b (\text{speed})\, dt = \int_a^b |\vec{r}\,'(t)|\, dt$$

ex: $\vec{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$, $0 \le t \le 2\pi$

Find the arclength



velocity $= r'(t) = \langle -3\sin 3t,\ 3\cos 3t,\ 4 \rangle$

$$\text{Speed} = |\vec{r}\,'(t)| = \sqrt{(-3\sin 3t)^2 + (3\cos 3t)^2 + 4^2}$$

$$= \sqrt{9\sin^2 3t + 9\cos^2 3t + 16}$$

$$= \sqrt{9(\sin^2 3t + \cos^2 3t) + 16}$$

$$= \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\text{arclength} = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} 5\, dt = 5t \Big|_0^{2\pi} = 10\pi$$

Defn: The arc length parameter of a curve is

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du = \text{length of a curve from } t=a \text{ to } t=t.$$

$$= \text{"odometer reading".}$$

So $\frac{ds}{dt} = |\vec{r}\,'(t)| = \text{speed}$ (by the Fund. Thm. of Calculus)

that is $\frac{d}{dt}(\text{odometer}) = \text{speedometer}$

Defn: (last time) unit tangent vector $= \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{r}\,'(t)}{ds/dt}$

so $\vec{r}\,'(t) = \left(\frac{ds}{dt}\right)\vec{T}$

Fact: $\vec{T}' \cdot \vec{T} = 0$ that is $\vec{T}' = \frac{d\vec{T}}{dt}$ and $\vec{T}$ are orthogonal

proof: $\vec{T}\cdot\vec{T} = |\vec{T}|^2 = 1$

$$0 = \frac{d}{dt}[1] = \frac{d}{dt}\left[\vec{T}\cdot\vec{T}\right] = \frac{dT}{dt}\cdot T + T\cdot\frac{dT}{dt}$$

$$= 2\left(\frac{dT}{dt}\cdot T\right) \Rightarrow \frac{0}{2} = \frac{d\vec{T}}{dt}\cdot\vec{T}.$$

Defn: $\dfrac{d\vec{T}}{ds} = \kappa\vec{N}$ where $|\vec{N}| = 1$. This defines both

κ = curvature and $\vec{N}$ = unit normal vector

because $N = \dfrac{dT/ds}{|dT/ds|}$ and $\kappa = \left|\dfrac{dT}{ds}\right|$

NOTE: $T\cdot T = N\cdot N = 1$ and $\vec{T}\cdot\vec{N} = 0$

by the "Fact" above.