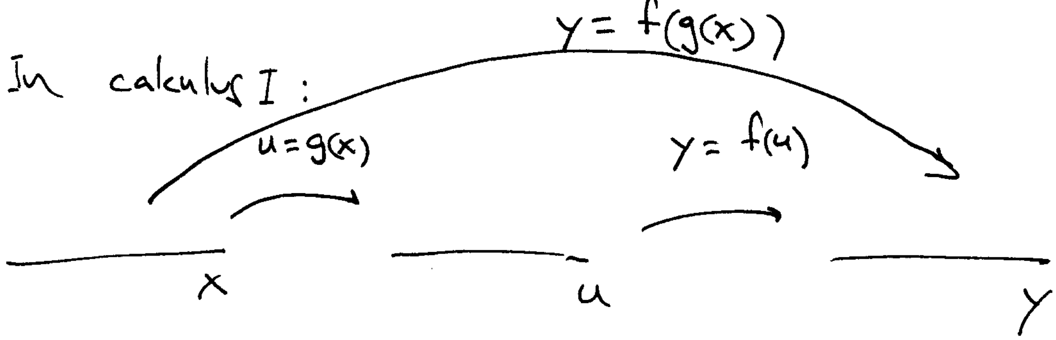
14.5 Chain Rules

In calculus I:

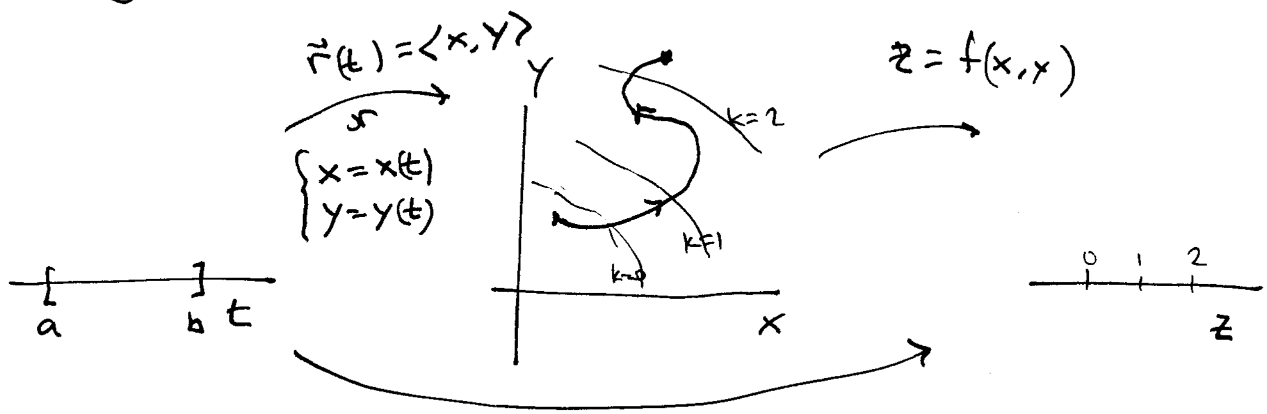


$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\Big[f(g(x))\Big] = \frac{d}{du}\big[f(u)\big]\cdot\frac{d}{dx}\big[g(x)\big]$$

$$= f'(u)\cdot g'(x)$$

$$= f'(g(x))\cdot g'(x)$$

A generalization:



$$z = f(\vec{r}(t)) \quad \text{or} \quad z = f(x(t), y(t))$$

<u>Chain Rule</u>: If $z = f(x,y)$ is differentiable, and $x = x(t)$, $y = y(t)$ are differentiable, then z is a differentiable function of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

<u>Idea</u> of the proof: $\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$

Divide by Δt: $\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} \to \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Remark: 1) The chain rule in <u>matrix notation</u> is

$$\left[\frac{dz}{dt}\right] = \left[\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}\right]\begin{bmatrix}\frac{dx}{dt}\\ \\ \frac{dy}{dt}\end{bmatrix}$$

2) In Dot product notation is

$$\frac{dz}{dt} = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$$

$$= \nabla f(x,y) \cdot \vec{r}\,'(t)$$

$$= |\nabla f(x,y)|\ |\vec{r}\,'(t)| \cos\theta$$

where $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$ = "the <u>gradient</u> of f at (x,y)"

14.5 2) Find $\frac{dz}{dt}$ if $z = \cos(x+4y)$ and $x = 5t^4$ and $y = t^{-1}$.

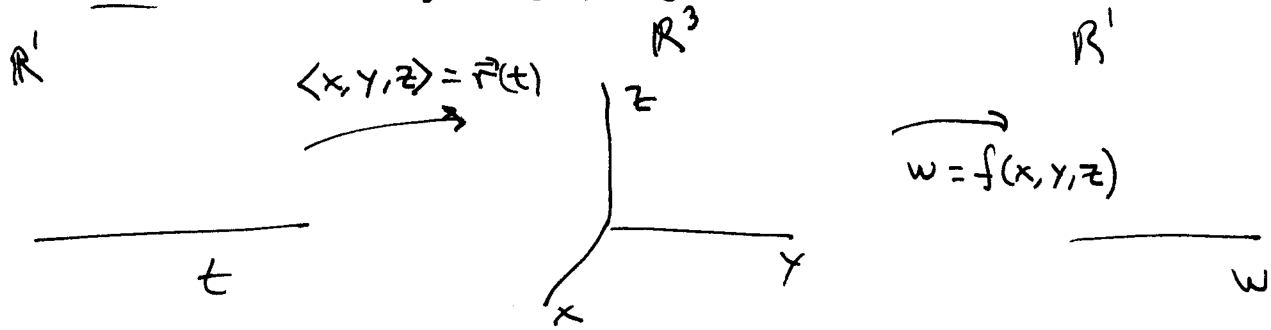
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= [\sin(x+4y)](20t^3) + [-4\sin(x+4y)](-t^{-2})$$

Now express everything in terms of t:

$$= [-\sin(5t^4+4t^{-1})](20t^3) + [-4\sin(5t^4+4t^{-1})](-t^{-2})$$

Remark: In this scenario

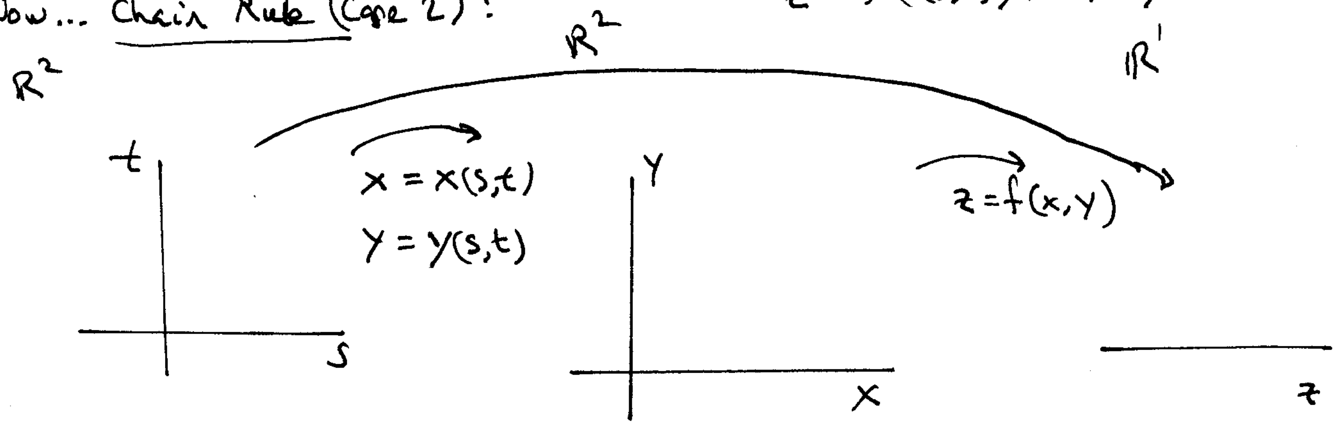


Chain Rule (case 1.5)

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

Now... Chain Rule (Case 2):

$z = f(x(s,t), y(s,t))$



Chain Rule:

$$\begin{cases} \dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} \\[2ex] \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} \end{cases}$$

matrix form:

$$\begin{bmatrix} \dfrac{\partial z}{\partial s} & \dfrac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\[2ex] \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{bmatrix}$$

14.5 part of 12) Find $\frac{\partial z}{\partial s}$ if $z = \tan\left(\frac{u}{v}\right)$ and $u = 2s + 3t$
$v = 3s - 2t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s}$$

$$= \left[\frac{1}{v}\sec^2\left(\frac{u}{v}\right)\right](2) + \left[\frac{-u}{v^2}\sec^2\left(\frac{u}{v}\right)\right](3)$$

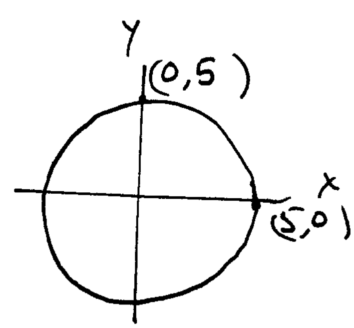
Now express in terms of s and t

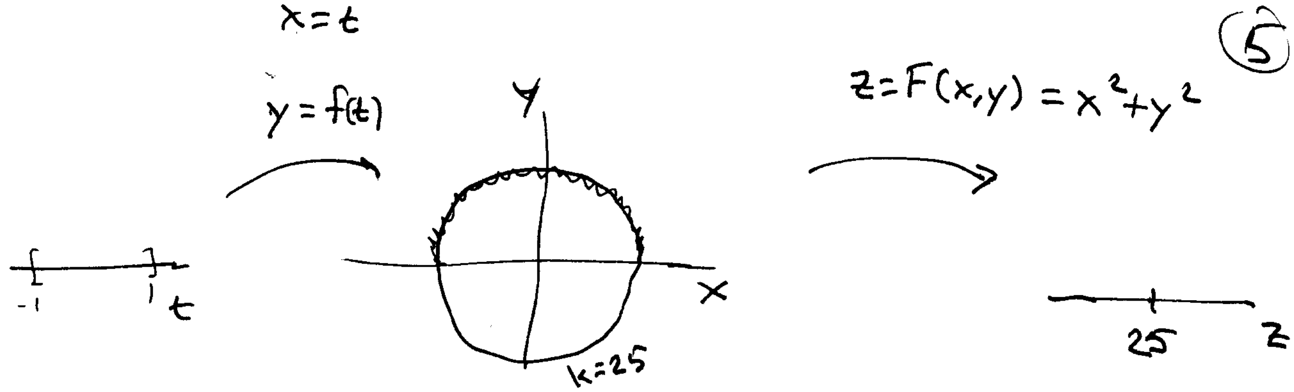
$$= \left(\frac{2}{3s-2t}\right)\sec^2\left(\frac{2s+3t}{3s-2t}\right) - \frac{3(2s+3t)}{(3s-2t)^2}\sec^2\left(\frac{2s+3t}{3s-2t}\right)$$

Implicit Differentiation

Defn: we say that $y = f(x)$ is implicitly defined if its graph is a subset of a level curve of $F(x,y)$.

ex: $x^2 + y^2 = 25$





<u>Remark</u>: Pretend we don't know this, but $f(t) = \sqrt{25-t^2}$.
y

In a more typical implicit differentiation problem, we <u>won't</u> have an equation for $y=f(x)$.

Recall the chain rule: $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

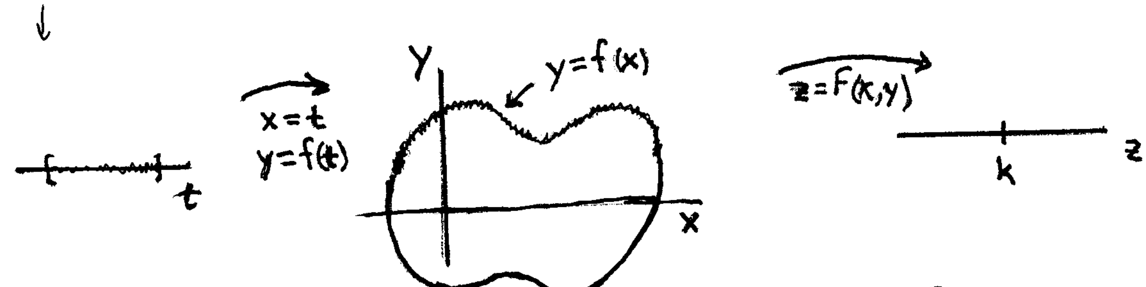
But $z = F(t, f(t)) = 25$ so

$$0 = \frac{d[25]}{dt} = F_x(x,y)\,\cancel{\frac{dt}{dt}}^{\,1} + F_y(x,y)\,\frac{dy}{dx}$$ * using that $x=t$

so $$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

that is, $$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Added after class ended. ↓



More generally, if $y=f(x)$ is implicitly defined by $F(x,y)=k$ so that so that $F(t, f(t)) = k$, by applying the chain Rule we get

$$0 = \frac{d[k]}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = F_x(x,y)\cdot\frac{dt}{dt} + F_y(x,y)\frac{dy}{dt} = F_x + F_y\cdot\frac{dy}{dx}$$ using $x=t$

$$\Rightarrow\quad F_y\cdot\frac{dy}{dx} = -F_x \quad\Rightarrow\quad \boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

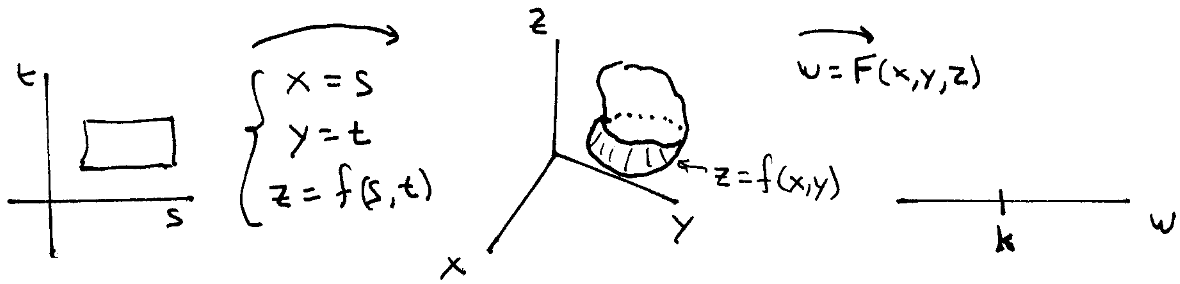
ex [see §2.6 Ex 2, p 159] Find $\frac{dy}{dx}$ if $y=f(x)$ is implicitly defined by $x^3+y^3=6xy$, or equivalently, $x^3+y^3-6xy=0$.

Let $F(x,y) = x^3+y^3-6xy$.

then $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2-6y}{3y^2-6x} = -\frac{3(x^2-2y)}{3(y^2-2x)} = \frac{2y-x^2}{y^2-2x}$

<u>Defn</u>: We say $f(x,y)$ is <u>implicitly</u> <u>defined</u> by the equation $w=F(x,y,z)=k$ if the surface $z=f(x,y)$ is a subset of the level surface of F corresponding to $w=k$.

That is, the following composition is a <u>constant</u> function of s and t: $F(s,t,f(s,t))=k$.



By the Chain Rule: $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$

~~Letting~~ noting that if $x=s$ then $\frac{\partial x}{\partial s} = \frac{\partial s}{\partial s} = 1$

and if $y=t$ then $\frac{\partial y}{\partial s} = \frac{\partial t}{\partial s} = 0$,

using that $x=s$

$$0 = \frac{\partial}{\partial s}[k] = \frac{\partial w}{\partial x}\cdot 1 + \frac{\partial w}{\partial y}\cdot 0 + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} = F_x(x,y,z) + F_z(x,y,z)\cdot\frac{\partial z}{\partial x}$$

$$\Rightarrow F_z\cdot\frac{\partial z}{\partial x} = -F_x \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

In a similar way, we get: $\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$

§14.5 #32) Suppose $z=f(x,y)$ is implicitly defined by the hyperboloid of one sheet, $x^2-y^2+z^2-2z=4$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Let $F(x,y,z) = x^2-y^2+z^2-2z$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2z-2} = \boxed{-\frac{x}{z-1}} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(-2y)}{2z-2} = \boxed{\frac{y}{z-1}}$$