15.3 Double integrals over non-rectangular regions

52) [All three "skills" in one problem.] Evaluate by <u>reversing</u> the order of integration.

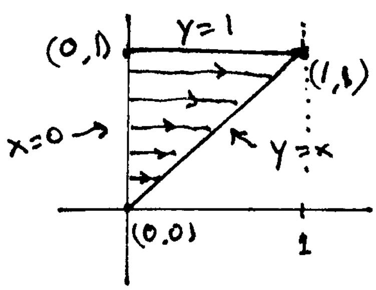
$$\int_0^1 \int_x^1 e^{\frac{x}{y}}\, dy\, dx$$

[Skill #3: what is the region of integration, D?]

Boundaries of D: $\left.\begin{array}{l} y = x \\ y = 1 \end{array}\right\}$ $x \le y \le 1$

$x = 0$

$x = 1$



[Skill #2: Look at D and set up an iterated integral as a type 2 region.]

$$\int_{y=0}^{y=1} \int_{x=0}^{x=y} e^{\frac{x}{y}}\, dx\, dy = \int_0^1 y e^{\frac{x}{y}} \Big|_{x=0}^{y} dy$$

[Skill #3: Do the iterated integral.]

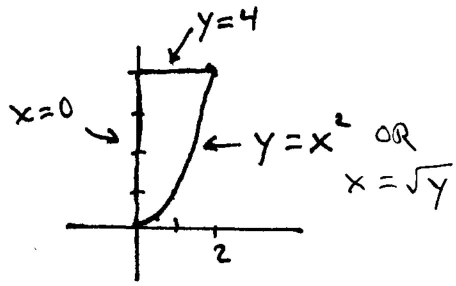
$$= \int_0^1 \left(y e^{\frac{y}{y}} - y e^{\frac{0}{y}} \right) dy = \int_0^1 (y e^1 - y e^0)\, dy$$

$$= \int_0^1 y(e-1)\, dy = (e-1)\int_0^1 y\, dy = (e-1)\frac{y^2}{2}\Big]_0^1 = \frac{(e-1)}{2}$$

[Exercises in Skills #2 and #3]

44) $$\int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx$$

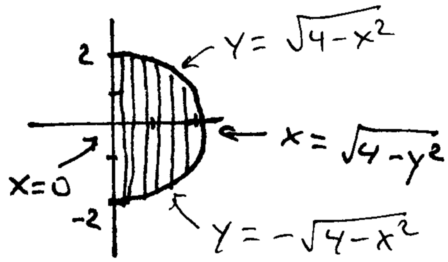
Boundaries of D: $y = x^2$, $y = 4$, $x = 0$, $x = 2$.



$$\int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$

46) $$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$

Boundaries of D: $x = 0$, $x = \sqrt{4-y^2}$, $y = -2$, $y = 2$



OR $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

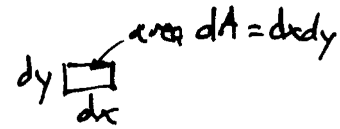
15.4 Double integrals in polar coordinates

<u>Remark</u>: Two ways to think about double integrals

(1) $\iint_D f(x,y)\,dA$

= volume of the solid <u>under</u> $z = f(x,y)$, over D.

(2) dA = element of <u>area</u>

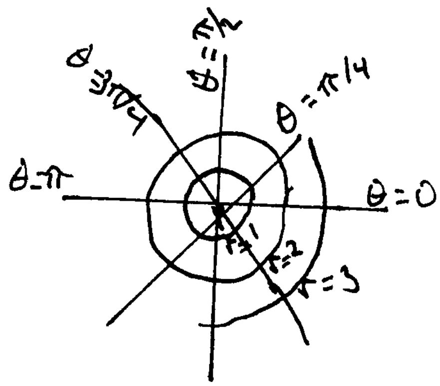


$f(x,y)$ = a weight to apply to that element of area

$f(x,y)\,dA$ = weighted element of area

$\iint_D f(x,y)\,dA$ = total ~~mass~~ of the weighted area elements.

$f(x,y)$: g/cm^2 $\quad dA$: cm^2 $\quad = g$

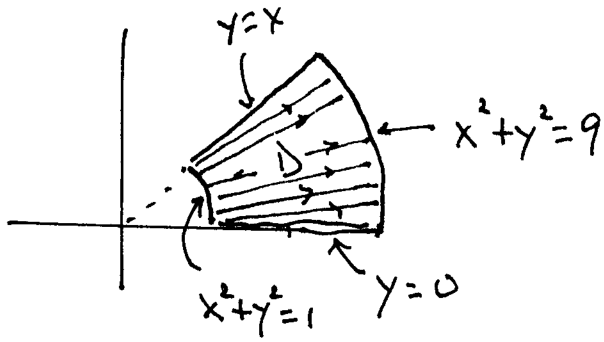


$$\boxed{\begin{array}{ll} x = r\cos\theta & x^2+y^2=r^2 \\ y = r\sin\theta & \dfrac{y}{x} = \tan\theta \end{array}}$$

<u>Also</u>: $dA = dx\,dy = r\,dr\,d\theta$

ex: Use polar coordinates to evaluate

$$\iint_D y \, dA$$ where D is this region:



Boundaries of D:

Rectangular		Polar
$x^2+y^2=9$	$\Leftrightarrow$	$r^2=9$ or $r=3$
$x^2+y^2=1$	$\Leftrightarrow$	$r^2=1$ or $r=1$
$y=0$	$\Rightarrow \frac{y}{x}=0 \Rightarrow \tan\theta=0 \Rightarrow \theta=0$	
$y=x$	$\Rightarrow \frac{y}{x}=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\pi/4$	

$$\int_0^{\pi/4}\int_1^3 (r\sin\theta)\, r\,dr\,d\theta = \int_0^{\pi/4}\int_1^3 r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \sin\theta \int_1^3 r^2\,dr\,d\theta$$

$$= \int_1^3 r^2\,dr \cdot \int_0^{\pi/4} \sin\theta\,d\theta$$

$$= \left[\frac{r^3}{3}\right]_1^3 \left[-\cos\theta\right]_0^{\pi/4} = \left[\frac{27}{3}-\frac{1}{3}\right]\left[-\frac{\sqrt{2}}{2}+1\right]$$

$$= \boxed{\frac{26}{3}\left(1-\frac{\sqrt{2}}{2}\right)} \approx 2.5384$$