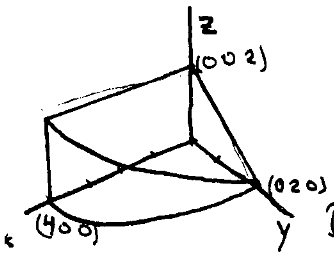
Warm-up

15.7 28) Sketch the solid whose volume is

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$$



Boundaries of E: $x = 4 - y^2$

$x = 0$

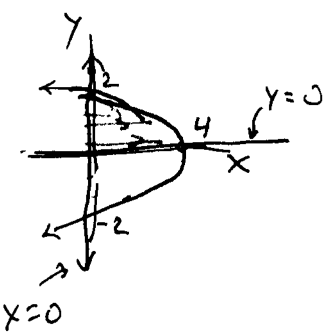
$z = 2 - y$ = plane

$z = 0$ = xy-plane

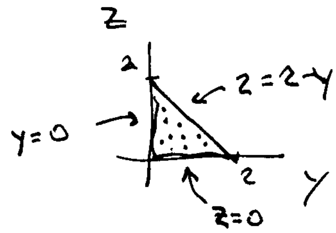
$y = 2$

$y = 0$

View along z-axis



View along x-axis



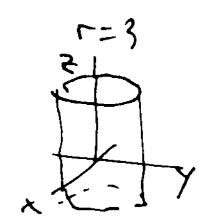
15.8 Triple integrals in cylindrical coordinates

Rectangular		Cylindrical	Also
x	=	$r\cos\theta$	$x^2+y^2=r^2$
y	=	$r\sin\theta$	$\frac{y}{x}=\tan\theta$
z	=	z	

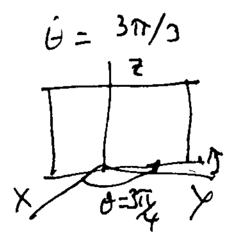
"element of volume" $= dV = dx\,dy\,dz = r\,dz\,dr\,d\theta$

A surface of constant...

r = constant is a cylinder of radius constant and axis = z-axis



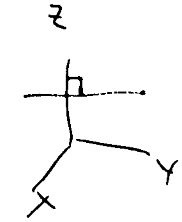
θ = constant is a plane containing the z-axis



z = constant is a plane parallel to the xy-plane

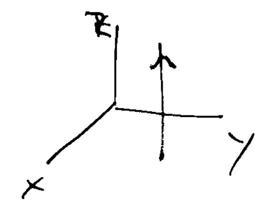
<u>If we vary only ...</u>

r (keep θ and z constant) we get a line intersecting the z-axis at right angles



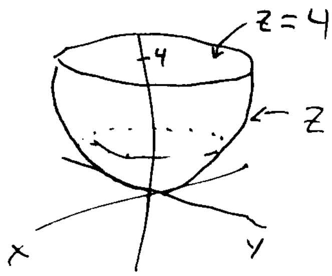
θ (keep r and z constant) circle with center on the z-axis, parallel to xy-plane

z (keep r and θ constant) we get a line <u>parallel</u> to the z-axis



18) [Set-up only] Evaluate $\iiint_E z\, dV$ where E is enclosed by the paraboloid $z = x^2+y^2$ and plane $z=4$.

$z=4$



$\leftarrow z = x^2+y^2$ OR
$z=r^2$ OR
$r=\sqrt{z}$

where do the surfaces intersect?

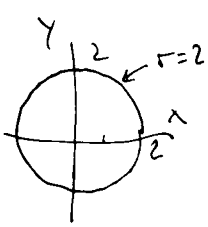
$$\begin{cases} z = r^2 \\ z = 4 \end{cases}$$

so $r^2=4$ where they intersect
or $r=2$ <u>and</u> $z=4$

<u>Set-up #1</u>

$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} z\, r\, dz\, dr\, d\theta =$$

<u>Top view</u>



<u>Set-up #2</u>

$$\int_0^{2\pi}\int_0^{4}\int_0^{\sqrt{z}} z\, r\, dr\, dz\, d\theta$$

30) Evaluate by changing to cylindrical coordinates

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx = (*)$$

Boundary of E:

$z = 9-x^2-y^2 \Leftrightarrow z = 9-r^2$

$z = 0$

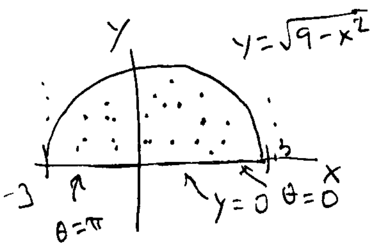
$y = \sqrt{9-x^2} \Leftrightarrow x^2+y^2=9 \Leftrightarrow r=3$

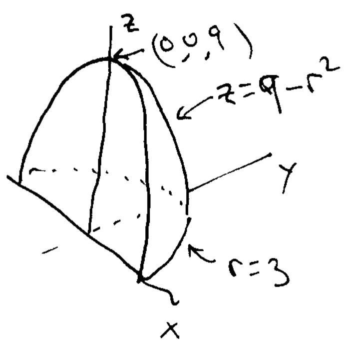
$y = 0 \Leftrightarrow r\sin\theta = 0 \Leftrightarrow \theta = 0$

$x = 3$

$x = -3$

Top view





Note: $\sqrt{x^2+y^2} = r$

$$(*) = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r\cdot r\,dz\,dr\,d\theta$$

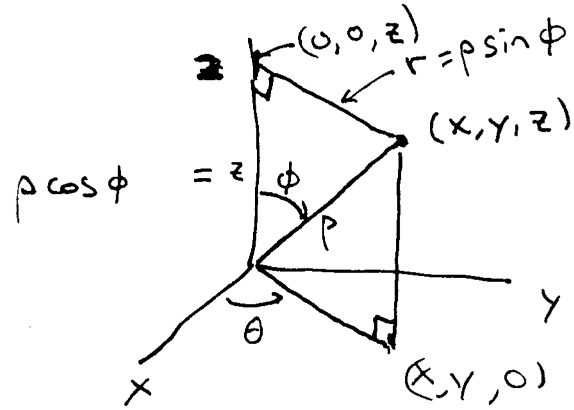
$$= \int_0^{\pi}\int_0^{3} r^2 z\Big]_0^{9-r^2}\,dr\,d\theta$$

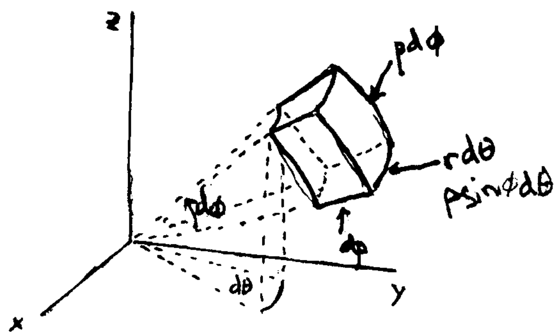
$$= \int_0^{\pi}\int_0^{3} r^2(9-r^2)\,dr\,d\theta = \int_0^{\pi}d\theta\cdot\int_0^{3}(-r^4+9r^2)\,dr$$

$$= \Big[\theta\Big]_0^{\pi}\Big[-\frac{r^5}{5}+3r^3\Big]_0^{3} = (\pi)\left(-\frac{3^5}{5}+3^4\right)$$

$$= (\pi)\cdot 3^4\left(-\frac{3}{5}+1\right) = (\pi)(81)\left(\frac{2}{5}\right) = \frac{162\pi}{5}$$

15.9 Triple integrals in spherical coordinates





Note: $r = \rho \sin\phi$

Rectangular		Cylindrical		Spherical
x	$=$	$r\cos\theta$	$=$	$\rho\sin\phi\cos\theta$
y	$=$	$r\sin\theta$	$=$	$\rho\sin\phi\sin\theta$
z	$=$	z	$=$	$\rho\cos\phi$
$dV = dx\,dy\,dz$	$=$	$r\,dz\,dr\,d\theta$	$=$	$\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Typically $0 \le \rho$; $0 \le \theta \le 2\pi$; $0 \le \phi \le \pi$

Note:

$$dV = (d\rho)(\rho\,d\phi)(\rho\sin\phi\,d\theta)$$
$$= \rho^2 \sin\theta\, d\rho\, d\phi\, d\theta$$

24) [Set-up only] Evaluate $\iiint_E y^2\,dV$ where E is the solid hemisphere

$$x^2+y^2+z^2 \le 9 \quad \text{and} \quad y \ge 0$$

other identities that may be useful:

$$\rho^2 = x^2+y^2+z^2 \qquad \frac{y}{x} = \tan\theta$$

$$\rho^2\sin^2\phi = x^2+y^2$$

$$\frac{\sqrt{x^2+y^2}}{z} = \frac{r}{z} = \frac{\rho\sin\phi}{\rho\cos\phi} = \tan\phi$$

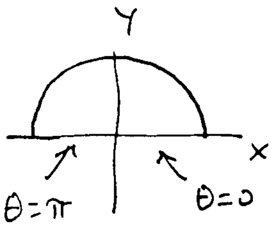
$$\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{\sqrt{r^2+z^2}} = \frac{\rho\cos\phi}{\rho} = \cos\phi$$

24 cont'd)

Boundaries of E:

$$x^2+y^2+z^2=9 \iff \rho^2=9 \iff \rho=3$$

$$y=0 \iff \rho\sin\theta=0 \iff \theta=0 \text{ or } \pi$$



$$\iiint_E y^2\,dV = \int_0^{\pi}\int_0^{\pi}\int_0^{3} (\rho\sin\phi\sin\theta)^2 \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi}\int_0^{3} \rho^4 \sin^3\phi \sin^2\theta \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{3}\rho^4\,d\rho \cdot \int_0^{\pi}\sin^3\phi\,d\phi \cdot \int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \int_0^{3}\rho^4\,d\rho \cdot \int_0^{\pi}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta \cdot \int_0^{\pi}(1-\cos^2\phi)\overbrace{\sin\phi\,d\phi}^{-d(\cos\phi)}$$

$$= \left[\frac{\rho^5}{5}\right]_0^3 \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi} \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi}$$

$$= \left[\frac{3^5}{5}\right]\left[\frac{\pi}{2}\right]\left[\left(-\cos\pi+\tfrac{1}{3}\cos^3\pi\right)-\left(-\cos 0+\tfrac{1}{3}\cos^3 0\right)\right]$$

$$= \frac{243\pi}{10}\left[\left(\tfrac{2}{3}\right)-\left(-\tfrac{2}{3}\right)\right] = \left(\frac{243\pi}{10}\right)\left(\frac{4}{3}\right) = \boxed{\frac{162\pi}{5}} = 32.4\pi$$

$$\approx 101.79$$