Another application of the Pigeon Hole Principle

This type of problems involves scheduling activities over a period of time. Suppose you have to finish writing 20 articles over a period of 14 days, with at least one article written a day. Then no matter how you schedule your work, there will be a period of consecutive days during which you write exactly 7 articles.

To prove that this is true, let x_i be the total number of articles you finish writing by the end of the i-th day, for $i = 1, 2, 3, \cdots, 14$. Since at least one article is written each day, we have

$$x_1 < x_2 < x_3 < \cdots < x_{13} < x_{14}$$

showing that all the x_i's are distinct whole numbers.

Adding 7 throughout gives

$$x_1 + 7 < x_2 + 7 < x_3 + 7 < \cdots < x_{13} + 7 < x_{14} + 7$$

showing that these 14 numbers are also distinct.

We also know that x_{14} must be equal to 20, and $x_{14} + 7$ must be equal to 27, therefore all the 28 numbers below must be less than or equal to 27.

$$1 \leqslant x_1,\ x_2,\ \cdots,\ x_{14},\ x_1 + 7,\ x_2 + 7, \cdots,\ x_{14} + 7 \leqslant 27$$

In other words, we have 28 pigeons, namely

$$x_1,\ x_2,\ \cdots,\ x_{14},\ x_1 + 7,\ x_2 + 7, \cdots,\ x_{14} + 7$$

living in 27 pigeon holes, namely

$$1,\ 2,\ 3,\ \cdots,\ 27$$

which implies that at least two of them must be sharing the same hole.

As we mentioned before, the first 14 'pigeon's are distinct and will be in different pigeon holes. Similarly no two of the last 14 'pigeon's will share a pigeon hole. The only possibility is that one of the first 14 'pigeon's shares a pigeon hole with one pigeon from the second half. More precisely, we have

$$x_i = x_j + 7$$

for some i and j from the set {1, 2, 3, ... , 14}. In other words,

$$x_i - x_j = 7$$

which means that 7 articles are written totally from the ($i + 1$)th day to the j-th day. $\dashv$

Exercise

An amateur photographer goes on a six week holiday with 60 rolls of film, and is certain to use at least one roll every day. Show that there is a period of consecutive days in which the photographer shoots exactly 23 rolls of film.

Another challenging problem in pigeon-hole principle

Five kids are sneaking into a orchard full of pear trees, hoping to steal some juicy pears. If the trees in the orchard are arranged in a square lattice (as hown below) and no two kids would like to share the same tree, prove that no matter how big the orchard is and no matter which trees they pick, there must be two kids whose line of sight is blocked by a tree in between.