Borel Isomorphism Relations of Countable Reduced Abelian p-Groups.

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Abstract. This paper covers two major results. The first one states that any algorithm that can determine whether two arbitrarily given countable reduced 2-groups are isomorphic is as complicated as the process of computing their Ulm invariants, namely, it has to go through a transfinite iteration of unbounded countable length. In the language of descriptive set theory, this can be stated precisely as "the set $\{(G_1, G_2) : G_1, G_2$ are isomorphic reduced 2-groups$\}$ is relatively Σ^1_1 to the set $\{(G_1, G_2) : G_1, G_2$ are reduced 2-groups$\}$ but is not relatively Borel".

The second theorem denies the possibility of finding a Borel process to construct isomorphisms between any two given isomorphic countable reduced p-groups.

Introduction

H. Ulm proved in 1933 that the structure of a reduced countable Abelian p-group is completely determined up to isomorphism by a sequence of invariants called the Ulm invariants. The original methods he invented for the computation of these invariants and the construction of isomorphisms require a transfinite iteration whose length, depending on the group, can be any arbitrarily large countable ordinal. One may therefore ask whether there is an alternative algorithm that requires only transfinite recursions with bounded countable lengths. More precisely, if each countable p-group is coded by an element in the Cantor space, can we find a Borel partial function from the Cantor space into itself that would compute the rank and Ulm invariants of any reduced countable Abelian p-group? Can we find a Borel procedure that can construct an isomorphism between any two given isomorphic reduced countable Abelian p-groups? In this paper, we prove that the answers to both questions are unfortunately negative.

We shall start our investigation with the search for a minimal substructure of a p-group that generates the whole group and also retains the characteristics of the group. Unless otherwise stated, all groups in this paper are assumed to be Abelian and the group operation is addition.

1. Some basic definitions and preliminary results

De...nition 1.1. A group G is a torsion group if all its elements have ...nite order.

A torsion group G is primary if, for a certain prime p, every element has order a power of p. In this case we also say that G is a p-group.

Theorem 1.2. Every torsion group is a direct sum of primary groups.

A proof of this theorem can be found in [6, p.5].

In the proof of the above theorem, we can see that G is in fact the unique direct sum of the G_p's where

$$G_p = \{g \in G : o(g) = p^k \text{ for some } k > 0\} \quad (1)$$

If G and H are isomorphic and $\varphi : G \to H$ is an isomorphism, then G_p must be isomorphic to H_p for every prime p and $\varphi \restriction G_p$ will be an isomorphism between them. We therefore shall only consider p-groups and their isomorphism relations from now on.

De...nition 1.3. A group G is divisible if for every x in G and every non-zero integer n there is an element y in G with ny = x.

De...nition 1.4. A p-group G is divisible if and only if for every x in G, there exists an element y in G with py = x.

The following lemma is well known and the proofs for the following two theorems can be found in [6].

Theorem 1.5. A divisible group is a direct sum of groups each isomorphic to the additive group of rational numbers or to $Z(p^\infty)$ (for various primes p).

Theorem 1.6. Any group G has a unique largest divisible subgroup M and $G = M \oplus N$ where N has no (non-zero) divisible subgroups.

De...nition 1.7. A group is reduced if it has no (non-zero) divisible subgroup.

2. Trees for p-groups

It is well known that a vector space can be generated by a basis which consists of independent elements. For p-groups, we can also ...nd some similar minimal generating subsets which will be called generating trees.

Throughout this paper, a tree is a partially order set in which the set of predecessors of any element in ...nite and linearly ordered.

Definition 2.1. A tree $(T_G; <)$ is a full tree of a p-group G if the underlying set T_G is the set of elements in G and for any $g, h \in T_G$, $g < h$ if $g \neq 0$ and there is a postive integer k such that $p^k g = h$. (The root of $(T_G; <)$ is the identity element.)

If G is reduced, or in other words, T_G has no infinite branch, then the rank of an element g in G is defined to be its rank in the full tree $(T_G; <)$, namely,

$$rk_G(g) = \sup\{rk_G(x) + 1 : x \in G \text{ and } x < g\} \tag{2}$$

The rank of G is the rank of the identity element in $(T_G; <)$.

Note: by abuse of notation, we identify T_G with $(T_G; <)$.

Definition 2.2. If T is a tree, we define G_T to be the formal p-group generated by the elements in T other than the root subjected to the relations $pb = a$; where b is an immediate successor of a in $T \setminus \{root\}$; and $pb = 0$ if b is an immediate successor of the root in T:

T is said to be well founded if it has no infinite branch, in this case G_T will be reduced.

A normal form for an element in G_T is a linear combination of distinct elements in T with nonzero coefficients in $\{0, 1, 2, \ldots, p-1\}$.

T is a subtree of T_G if T is a subset of T_G with the induced partial ordering and is closed under predecessors. In this case we write $T \leq T_G$ by abuse of notation. Suppose $T \subseteq T_G$ and $\phi : G_T \to G$ is the natural homomorphism. We say that T is non-redundant if ϕ is injective, T generates G if ϕ is surjective and T is a generating tree for G if ϕ is bijective.

T is said to be a nice generating tree if it is a generating tree and it splits at a node $g \in T$ only if g is the root or $\mathrm{rank}_T(g)$ is a limit ordinal.

Note: If T is a tree and G_T is the p-group generated by T, then T is canonically embeddable into $T_{(G_T)}$. And if T generates G, then G_T coincides with G.

Proposition 2.3. Let T be a tree. If G_T is the p-group generated by T, then the normal form for each element in G_T is unique.

Proof: Suppose that we have two different normal forms

$$a_1x_1 + \cdots + a_kx_k \text{ and } b_1y_1 + \cdots + b_\ell y_\ell \tag{3}$$

Without loss of generality, we may assume that $x_1, \cdots, x_k, y_1, \cdots, y_\ell \in T$ are all distinct. Assume further that x_1 has maximal order.

We define a homomorphism $\varphi : G_T \to Z$ such that

(1) $\varphi(x_1) = \frac{1}{p}$,

(2) $\varphi(x_2) = \cdots = \varphi(x_k) = \varphi(y_1) = \cdots = \varphi(x_\ell) = 0$

(3) if z is a generator such that $p^k = x_1$ for some $k > 0$, then $\varphi(z) = \frac{1}{p^{k+1}}$

(4) All other generators are sent to 0

We then have

$$\varphi(a_1x_1 + \cdots + a_kx_k) = \frac{a_1}{p} \tag{4}$$

and

$$\varphi(b_1y_1 + \cdots + b_\ell y_\ell) = 0 \tag{5}$$

which means that these normal forms cannot be equal. $\square$

In the case that T is a generating tree for G, the normal form of an element $g \in G$ with respect to T will be defined similarly and is also unique by the above proposition.

The following proposition tells us more about the beauty of normal forms.

Proposition 2.4. Suppose that $T \subseteq T_G$ is non-redundant and we have

(1) $a_1, \cdots, a_k \in \{1, \cdots, p-1\}$
(2) $x_1, \cdots, x_k \in T$
(3) $b_1, \cdots, b_\ell \in \{1, \cdots, p-1\}$
(4) $y_1, \cdots, y_\ell \in T$

and

$$a_1x_1 + \cdots + a_kx_k = b_1y_1 + \cdots + b_\ell y_\ell \tag{6}$$

If $a_1x_1 + \cdots + a_kx_k$ is in normal form (i.e. all $x_1, \cdots, x_k$ are distinct and non-zero), then there exist ρ_i's; $i = 1, \cdots, \ell$ and $0 \le \rho_i \le b_i$ with at least one ρ_i non-zero such that

$$a_1x_1 = \sum_{i \le \ell} \rho_i y_i \tag{7}$$

Proof: By induction on the number of steps in reducing $b_1y_1 + \cdots + b_\ell y_\ell$ to its normal form. Observe that any number $b \ge p$ can be written uniquely as

$$c_0 + c_1p + \cdots + c_mp^m \tag{8}$$

with $0 \le c_0, \cdots, c_m < p$, $m \ge 1$ and $c_m > 0$. $\square$

The rank function has certain nice properties as we can see in the following propositions which can be proved by transfinite induction on rank.

Proposition 2.5. Let G be a p-group and T_G be its full tree. We have the following properties:

(1) for every $x \in G$, if $x \neq 0$ and $rk_G(x) > 0$ then

$$|\{y : py = x\}| = |\{y \in G : py = 0\}| \qquad (9)$$

In other words, T_G is uniformly branching except possibly at the root.
(2) $rk_G(x) = rk_G(-x)$
(3) $rk_G(x + y) \geq \min\{rk_G(x); rk_G(y)\}$ and equality holds if $rk_G(x) \neq rk_G(y)$.
(4) If $G = H \oplus K$, $x \in H$, and $y \in K$ then $rk_G(x + y) = \min\{rk_H(x); rk_K(y)\}$.

Proposition 2.6. Assuming that $T \subseteq G$ is a tree and it generates G, then the following are equivalent:

(1) T is non-redundant,

(2) For every distinct non-zero $g_1; \cdots; g_k \in T$ and every integers $\lambda_1; \cdots; \lambda_k \in \{1; 2; \cdots; p-1\}$, we have $rk_G(\lambda_1 g_1 + \cdots + \lambda_k g_k) = \min\{rk_G(g_i) : i \leq k\}$

(3) For every distinct non-zero $g_1; \cdots; g_k \in T$ and any integers $\lambda_1; \cdots; \lambda_k \in \{1; 2; \cdots; p-1\}$, if $rk_G(g_1) = \cdots = rk_G(g_k) = \alpha$, then $rk_G(\lambda_1 g_1 + \cdots + \lambda_k g_k) = \alpha$

If any one of the above is true then we have $rk_G(g) = rk_T(g)$ for all $g \in T$.

Note: In the above proposition, even if we drop the hypothesis that T generates G, we still have (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1).

Definition 2.7. If $X \subseteq G$ satisfies either condition (2) or (3) in the above proposition, then we say that X is rank independent.

Note: The above proposition implies that a generating tree of G is always rank independent. On the other hand, a maximal rank independent subtree of G_T may not be a generating tree for G, as we shall see in an example coming shortly afterwards, but nevertheless we have the following lemma.

Lemma 2.8. Let G be a countable reduced p-group, T_G be its full tree. If T is a subtree of T_G satisfying
(1) $\forall a \in T$, $rk_T(a) = 0$ implies $rk_G(a) = 0$,
(2) T is rank independent,
(3) T generates all order p elements of G
then T is a generating tree for G.

Proof: By proposition 8, it suffices to show that T generates G. Let's induct on the order of elements in G.

(i) $o(h) = p$: hypothesis.

(ii) $o(h) = p^l$, $l > 1$:

By the induction hypothesis, T generates ph and so there are $g_1; \cdots; g_k \in T$ and $\lambda_1; \cdots; \lambda_k \in Z_p \setminus \{0\}$ such that

$$ph = \lambda_1 g_1 + \cdots + \lambda_k g_k \tag{10}$$

Since T is rank independent and $rk_G(ph) > 0$, $rk_G(g_i) = rk_G(\lambda_i g_i) > 0$ for all $i \leq k$. From the given condition (1), this implies that $rk_T(g_i) > 0$ for all $i \leq k$. For each $i \leq k$, let's pick a $t_i \in T$ such that $pt_i = g_i$ and let $s = \lambda_1 t_1 + \cdots + \lambda_k t_k$. Then $s - h$ has order p and s is generated by T, hence h is also generated by T.

$\square$

Using axiom of choice, we can prove that every vector space has a basis. But the situation for p-groups is quite different; one can prove the existence of generating trees only in special cases, such as when the group has finite rank (see below) or the group is countable. The proof for the latter case is much more difficult and will not be given until we have developed enough machinery in section 3.

Theorem 2.9. (Using Axiom of Choice) Every p-group of finite rank has a nice generating tree.

Proof: Let G be a p-group of rank $n + 1$. For every $i \leq n$, let

$$A_i = \{g \in G : o(g) = p \text{ and } rk(g) = i\} \tag{11}$$

and for each $g \in G$ with $o(g) = p$, let us choose, by the axiom of choice, a path P_g starting at g with maximal length.

Note that if $h \in P_g$ and $ph \neq 0$, then $rk(ph) = rk(h) + 1$.
We shall build T_G as the union of subtrees T_k, $k = 1; 2; \cdots; n$, where each T_k is constructed by the following procedure:

Using Zorn's lemma, choose a maximal subset B_k of A_k satisfying the following condition:

$$\forall g_1; \cdots; g_m \in B_k; a_1; \cdots; a_m \in Z_p;\ rk(a_1 g_1 + \cdots + a_m g_m) = k \tag{12}$$

T_k is then defined to be the union $(\bigcup_{g \in B_k} P_g) \cup \{0\}$. Clearly T_G is non-splitting except at the root and it is not difficult to prove that T is also rank independent, hence by lemma 2.8 T_G is a nice generating tree.

$\square$

Definition 2.10. [Ulm invariants] Let G be a reduced p-group, for each ordinal $\alpha < \omega_1$ we define

$$G^{\alpha} = \{g \in G : o(g) = p \text{ and } rk_G(g) \geq \alpha\} \tag{13}$$

The αth Ulm invariant of G, $Ulm_G(\alpha)$, is defined to be the dimension of the vector space $G^{\alpha}/G^{\alpha+1}$ over the field Z_p.

Definition 2.11. The Ulm-sequence of G is a function f_G whose domain is the rank of G and for every $\alpha < rk(G)$, $f_G(\alpha) = Ulm_G(\alpha)$.
If T is a well-founded tree, then we define the Ulm invariants and the Ulm-sequence of T to be those of the p-group generated by T.

The following proposition gives a direct procedure to calculate the Ulm invariants of nice well-founded trees.

Proposition 2.12. If T is a nice generating tree for G, then the αth Ulm invariant of G is the cardinality of the following set

$$\{a \in T : a \neq 0; rk_T(a) = \alpha \text{ and either } pa = 0 \text{ or } rk_T(pa) \text{ is a limit ordinal}\} \tag{14}$$

Proof: If $x \in G_{\alpha}$, let $[x]$ denote the equivalence class of x in $G_{\alpha}/G_{\alpha+1}$.
Let

$$\begin{aligned} A_{\alpha} &= \{a \in T : rk(a) = \alpha \text{ and } o(a) = p\} && (15)\\ B_{\alpha} &= \{x \in T : rk(x) = \alpha;\ o(x) > p \\ &\qquad \text{and } rk(px) \text{ is a limit ordinal}\} && (16) \end{aligned}$$

For each $x \in B_{\alpha}$, let's choose an element $g_x \in T$ such that $rk(g_x) > \alpha$ and $px = pg_x$.
We shall show that the set

$$D = \{[a] : a \in A_{\alpha}\} \cup \{[x - g_x] : x \in B_{\alpha}\} \tag{17}$$

forms a basis for $G_{\alpha}/G_{\alpha+1}$ over Z_p.
Clearly the above set is a subset of $G_{\alpha}/G_{\alpha+1}$ and it is linearly independent over Z_p because T is rank independent.
To show that every element $y \in G_{\alpha}$, $[y] \in G_{\alpha}/G_{\alpha+1}$ can be generated by the above set, it suffices to consider only those y whose normal form (in terms of elements in T) does not mention elements in A_{α}.

Claim If $x \in T$ appears in the normal form of y and $rk(x) = \alpha$, then $x \in B_{\alpha}$.

Proof: Elementary ⊣
Now let

$$y' = \sum \eta_x (x - g_x) \tag{18}$$

where summation is over the set of all $x \in B_\alpha$, x appears in the normal form of y and η_x is the coefficient of x in the normal form of y.

Obviously y' has order p and since T is rank independent, the claim implies that $y - y'$ has rank $> \alpha$. Therefore $y - y' \in G_{\alpha+1}$ and hence $[y] = [y']$ is generated by the set D. □

Lemma 2.13. Let G be a p-group of rank ω. If G has a generating tree then G has a nice generating tree.

Proof: If G has a generating tree, then G can be written as a direct sum of subgroups each of finite rank. By our previous result, any p-group of finite rank has a nice generating tree and so G is the direct sum of such groups. □

There is also a constuctive proof that we will not have space to include here.

Proposition 2.14. There is an uncountable p-group with rank ω which has no nice generating tree and hence no generating tree at all.

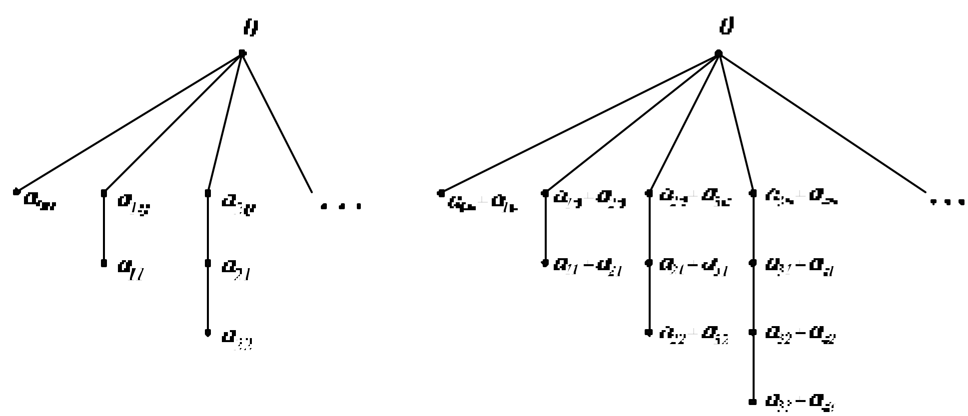
Proof: For each $i \in \omega \setminus \{0\}$, let H_n be a cyclic group of order p^n and let G' be the direct product of $\{H_i : i > 0\}$.

Our G would then be the torsion subgroup of G', or more explicitly

$$G = \{\langle h_1, h_2, \cdots \rangle : h_i \in H_i \text{ and } \exists k \in \omega \text{ such that } o(h_i) < p^k \text{ for all } i\} \tag{19}$$

It is easy to check that the Ulm invariants of G are all 1, so if G is a direct sum of cyclic groups then G would be countable. □

Example 2.15. A maximal rank independent subtree that is not a generating tree.



Let G be the 2-group generated by the nice tree in Figure 1.

The tree T_0 in the following figure is a maximal rank independent subtree of the full tree of G but it does not generate the element a_{00} and hence it cannot be a generating tree for G.

There is also a simple example of a minimal spanning tree that is not rank independent (see …gure 2 below). In other words, we cannot expect to get a generating tree by trimming any spanning tree, and the situation more complex than that in a vector space where any minimal generating set is automatically a maximal linearly independent set.

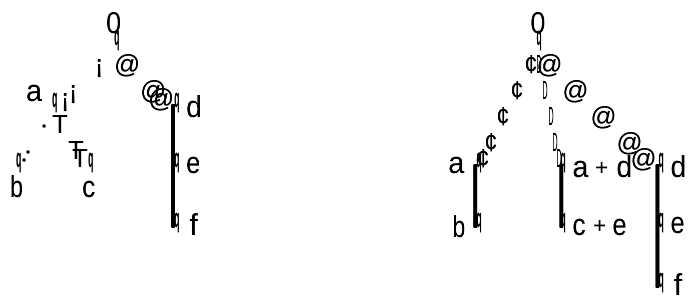


Figure 1:

3. Existence of nice generating trees

De…nition 3.1. Let G be a countable reduced 2-group such that its underlying set is a subset of the natural numbers. We shall code G by the sequence ®_G 2 $^!2$ in the following manner:

We …rst de…ne two sequences $\text{®}_1; \text{®}_2$ 2 $^!2$ by

$$\text{®}_1(n) = 1 \ \$ \ n \ 2 \ G \qquad (20)$$

$$\text{®}_2(\text{h}m; n; l\text{i}) = 1 \ \$ \ m; n; l \ 2 \ G \text{ and} \qquad (21)$$

$$m +_G n = l \qquad (22)$$

®_G is then constructed by merging ®_1 and ®_2. More precisely,

$$\text{®}_G(2n) = \text{®}_1(n) \qquad (23)$$

$$\text{®}_G(2n + 1) = \text{®}_2(n) \qquad (24)$$

Theorem 3.2. (Ulm's Theorem)

Two countable reduced p-groups are isomorphic if and only if they have the same Ulm invariants.

A proof of this theorem can be found in [6, p.26-30].

Definition 3.3. Let $f : \lambda \to \omega \cup \{\aleph_0\}$ be a function from a countable ordinal λ to the set of countable cardinals. We say that f is an Ulm-function if

(i) for every pair of limit ordinals α and β such that $\beta < \alpha \leq \lambda$, f takes non-zero values at infinitely many ordinals between α and β. (0 is also considered to be a limit ordinal here).

(ii) If $\lambda = \alpha + 1$ is a successor ordinal then $f(\alpha) \neq 0$.

Note: if f is an Ulm-function as defined above then

(1) for every limit ordinal $\alpha < \lambda$, $f \restriction \alpha$ is also an Ulm-function.

(2) for every limit ordinal $\alpha < \lambda$, the set of ordinals $\{\beta : f(\beta) \neq 0\}$ is unbounded in α.

Proposition 3.4. $f : \lambda \to \omega \cup \{\aleph_0\}$ is an Ulm-function if and only if there is a countable 2-group G whose Ulm-invariant sequence is exactly f, i.e. $rk(G) = \lambda$ and the α-th Ulm invariant of G is $f(\alpha)$ for all $\alpha < \lambda$.

Proof: The sufficiency follows directly from the definition of Ulm-invariants, while the necessity follows from theorem 3.7 and the fact that the 2-group generated by a nice tree T will have the same Ulm-invariant sequence as T. $\square$

Lemma 3.5. If λ is a limit ordinal and f is an Ulm-function with domain λ, then there is a sequence of Ulm-functions $\langle g_n : n \in \omega \rangle$ such that

(i) $dom(g_n) = \lambda$, for all $n \in \omega$

(ii) $f = \sum_{n \in \omega} g_n$

Proof:

For each $\gamma < \lambda$ which is 0 or a limit ordinal, partition the infinite set $\{m \in \omega : f(\gamma + m) > 0\}$ into infinitely many infinite sets $S_n^{(\gamma)}$; and let $g_n(\gamma + m)$ be $f(\gamma + m)$ if $m \in S_n^{(\gamma)}$, 0 otherwise. $\square$

Lemma 3.6. If λ is a limit ordinal and f is an Ulm-function with domain λ then there is a sequence of ordinals $\langle \alpha_n : n \in \omega \rangle$ cofinal in λ and a sequence of Ulm-functions $\langle g_n : n \in \omega \rangle$ such that

(i) $dom(g_n) = \alpha_n$ for all $n \in \omega$

(ii) $f = \sum_{n \in \omega} g_n$

Proof: If λ is a limit of limits, we choose a sequence of limit ordinals $\langle \alpha_m : m \in \omega \rangle$ cofinal in λ; split each set $\{\beta \in [\alpha_m, \alpha_{m+1}) : f(\beta) \neq 0\}$ into infinitely many infinite sets and use a method similar to the above lemma to construct the g_n 's

If $\lambda = \gamma + \omega$ for some limit ordinal λ; and $\langle \beta_n : n \in \omega \rangle$ is an increasing enumeration of the set $\{\beta \in [\gamma, \lambda) : f(\beta) \neq 0\}$; we then let $\alpha_n = \beta_n + 1$ and $g_n(\beta_n) = f(\beta_n)$; $g_n \restriction \gamma$ is then constructed by a method similar to that in the above lemma. $\square$

Theorem 3.7. For every Ulm-function $f : \lambda \to \omega \cup \{\aleph_0\}$, there is a well-founded nice tree T whose Ulm-sequence is f.

Proof: Again we shall use induction on λ.
(i) $\lambda < \omega$: Trivial.
(ii) λ is a limit ordinal:
By lemma 3.6, f can be written as a sum $\sum_{n\in\omega} f_n$ of Ulm-functions such that each f_n has domain some $\alpha_n < \lambda$. Therefore by induction assumption, there are nice trees T_n's such that the Ulm-sequence of T_n is exactly f_n. We can simply define T to be the amalgamation of all the T_n's at the root.
(iii) $\lambda (> \omega)$ is a successor:
Let $\lambda = \alpha + m$ where α is a limit ordinal and $m(> 0) \in \omega$. The restriction $g = f \restriction \alpha$ is then an Ulm-function with domain a limit ordinal and hence by lemma 3.5, g can be expressed as a sum $\sum_{n\in\omega} g_n$ where each g_n is an Ulm-function with domain α. By induction assumption on λ, we can find nice trees T_n's whose Ulm-sequences are g_n's. Let T^* be a nice tree of rank m and with exactly $f(\alpha + k)$ branches of length $k + 1$ for each $k < m$. Such a tree exists because by the definition of an Ulm-function, $f(\alpha + m - 1)$ is nonzero. Now we can construct T by attaching the T_n's to the leaves of T^* (i.e. the root of T_n is amalgamated with one leaf of T^*), such that

(i) At least one T_n is attached to each leaf of T^* and

(ii) Each T_n is attached to one and only one leaf of T^*.

This T works. □

Corollary 3.8. Every countable reduced 2-group has a nice generating tree.

Proof: Given any reduced 2-group G, let f be its Ulm-sequence. By the above theorem, we can find a nice tree T whose Ulm-sequence is also f. If we let $H(T)$ be the 2-group generated by T then G and $H(T)$ are isomorphic according to Ulm's theorem.

Suppose $\varphi : G \to H(T)$ is an isomorphism, then since T is a subset of $H(T)$ we can take its inverse image $\varphi^{-1}(T)$ which will be a generating tree for G. □

Proposition 3.9. Any two recursive rank one countable 2-groups are recursively isomorphic if and only if they have the same cardinality.

Proof: The necessity is obvious and to prove the sufficiency, let G and H be two such groups and enumerate their elements as

$$\{g_0, g_1, \cdots\} \text{ and } \{h_0, h_1, \cdots\} \tag{25}$$

We shall assume that $g_0 = h_0 = 0$.

Our isomorphism φ will be defined by recursion: $\varphi(g_0) = h_0$; and uppose that we have already defined φ on $\{g_0; g_1; \cdots; g_n\}$ such that $\varphi \upharpoonright \{g_0; \cdots; g_n\}$ is a finite monomorphism. We then consider the following two cases separately.

(1) g_{n+1} is generated by $\{g_0; \cdots; g_n\}$:

If $g_{n+1} = g_i + g_j + g_k$ for instance, we define $\varphi(g_{n+1}) = \varphi(g_i) + \varphi(g_j) + \varphi(g_k)$.

(2) g_{n+1} is independent of $\{g_0; \cdots; g_n\}$:

In this case we define $\varphi(g_{n+1})$ to be h_k where k is the smallest natural number such that h_k is not generated by the set $\{0; \varphi(g_1); \varphi(g_2); \cdots; \varphi(g_n)\}$:

$\Box$

Proposition 3.10. There are two recursive rank two countable 2-groups with recursive generating trees such that they have the same Ulm invariants but are not recursively isomorphic.

Proof: We shall construct G_1, G_2 such that their 1st and 2nd Ulm invariants are both $\aleph_0$.

Let G_1 be generated by the following recursive tree T_1;i.e. we attach one more node to each x_n if n is even (see figure 3).

It is then easy to see that the set $\{g \in G_1 : o(g) = 2 \text{ and } rk(g) > 0\}$ is recursive.

On the other hand, let G_2 be generated by the recursive tree T_2 with the following sets of generators

$\{t_n : n \in \omega\}$ and $A = \{z_{(n;n;m)} : \{n\}(n) \text{ terminates after exactly } m \text{ steps}\}$ (where $\{e\}(x)$ is the universal recursive function) and with the following relations:

$$\forall n \in \omega;\ 2t_n = 0 \tag{26}$$

$$\forall (n; n; m) \in A;\ 2z_{(n;n;m)} = t_n \tag{27}$$

It is also easy to see that the tree T_2 is recursive.

However, the set $\{t_n : n \in \omega \text{ and } rk(t_n) > 0\}$ is recursively enumerable but not recursive and therefore, G_2 cannot be recursively isomorphic to G_1. $\Box$

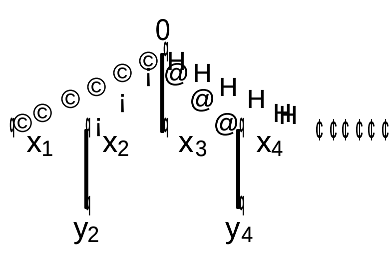


Figure 2:

Proposition 3.11. Every recursive countable 2-group of rank $\leq \omega$ has a Δ^0_2 nice generating tree.

Proof: If G is such a group, we shall show that G has a nice generating tree T which is recursive in an r.e. oracle, hence T is Δ^0_2.

The oracle $\alpha(\in {}^{\omega}2)$ is defined by

$$\alpha(\langle x; m\rangle) = 1 \leftrightarrow x \in G \;\&\; \exists y \in G\ 2^m y = x \tag{28}$$

The following relation and functions are easily seen to be recursive in this oracle α:

(1) $x \mapsto rk_G(x)$
(2) $\{(x; y) : x; y \in G \;\&\; rk(x) \geq rk(y)\}$
(3) $x \mapsto$ the first longest path below x (may be empty)

Before we proceed further, we need the following definitions.

Definition 3.12. Suppose $g \in G$ such that $rk(g)$ is a successor ordinal, then we say that $P = \langle a_0; a_1; \cdots; a_k\rangle$ is a path below g with maximal rank property if

(1) $2a_0 = g$ and $2a_{i+1} = a_i$ $\forall i < k$.
(2) $\forall i < k$, $rk(a_i)$ is a successor ordinal.
(3) $rk(a_i) = rk(a_{i+1}) + 1$ $\forall i < k$
(4) $rk(a_k)$ is either 0 or a limit ordinal.

Note: If $rk(x)$ is finite, then any longest path below x will have the maximal rank property.

Definition 3.13. A tree $T \subseteq G$ is said to be a nice potential generating tree for G if it satisfies the following conditions,

(1) for every $g \in T$, T splits at g only if $g = 0$ or $rk_G(g)$ is a limit ordinal.
(2) for every $g \in T$, $rk_T(g) = 0$ only if $rk_G(g)$ is either 0 or a limit ordinal.
(3) for every $g \in T$, if $2g \neq 0$ and $rk_G(2g)$ is not a limit ordinal, then $rk_G(2g) = rk_G(g) + 1$
(4) T is rank independent
(5) $\forall a; b$ (distinct) $\in T$, if $2a = 2b \neq 0$, then $rk_G(a) \neq rk_G(b)$.

Lemma 3.14. Let G be a 2-goup of rank $\leq \omega$, T be a nice potential generating tree for G and $x \in G \setminus T$ with order 2,

(a) If $\{x\} \cup \{t \in T : o(t) = 2\}$ is rank independent, then so is $\{x\} \cup T$.
(b) If $\{x\} \cup T$ is rank independent and P is a path of maximal rank below x (in particular, $P \not\subseteq T$), then $\{x\} \cup T \cup P$ is also rank independent. The proof of this lemma is straight forward and is left to the reader.

Main Construction (for proposition 3.11)

Let $c_0; c_1; c_2; \cdots$ be the list of all order 2 elements in G. We shall build T as the increasing union of finite nice potential generating trees $\{T_n : n \in \omega\}$ such that T_{n+1} generates c_n.

Stage 0: Let $T_0 = \{0\}$

Stage $n+1$: Suppose T_n has already been constructed.

Case (i) If c_n is generated by T_n, we define $T_{n+1} = T_n$.

Case (ii) If c_n is not generated by T_n and $\{c_n\} \cup T_n$ is still rank independent, we then define

$$T_{n+1} = T_n \cup \{c_n\} \cup \text{“ the first longest path below } c_n\text{”} \tag{29}$$

By the above lemma, T_{n+1} is still rank independent.

Case (iii) c_n is not generated by T_n but $\{c_n\} \cup T_n$ is rank dependent.

By the above lemma, $\{c_n\} \cup D_n$ must also be rank dependent where

$$D_n = \{t \in T_n : o(t) = 2\} \tag{30}$$

Let

$$m = \max\{rk(c_n + \sum_{j \in J} c_j) : J \subseteq \{i : c_i \in T_n\}\ \} \tag{31}$$

and let J_0 be the first subset of $\{i : c_i \in T_n\}$ such that $rk(c_n + \sum_{j \in J_0} c_j) = m$ (note that $n \notin J_0$). Since $c_n + \sum_{j \in J_0} c_j$ has order 2 and is not generated by T_n, there must be an $\ell > n$ such that

$$c_\ell = c_n + \sum_{j \in J_0} c_j \tag{32}$$

Moreover, $\{c_\ell\} \cup D_n$ is rank independent by the choice of J_0 and the definition of m, hence we can define

$$T_{n+1} = T_n \cup \{c_\ell\} \cup \text{“ the first longest path below } c_\ell\text{”} \tag{33}$$

so that T_{n+1} is still rank independent and generates c_n.

This guarantees that T satisfies condition (3) of lemma 2.8, and condition (1) is satisfied by the choice of a longest path below each c_n included in T. Finally, T is rank independent because $T = \bigcup_{n \in \omega} T_n$ and each T_n is. Therefore T is a generating tree and is recursive in α. $\square$

4. Negative results

Theorem 4.1. The set $R = \{G : G$ is a countable reduced 2-group$\}$ is strictly Π^1_1.

Proof: Let us consider the map $\varphi : R \to$ Ordinals defined by,

$$\varphi(G) = rk(G) \tag{34}$$

Since the rank of G is the same as the rank of its full tree T_G, we can rewrite φ as the composition of two maps

$$G \mapsto T_G \mapsto rk(T_G) \tag{35}$$

The first one is a continuous map and the latter is a well known Π^1_1- rank, hence φ is also a Π^1_1- rank. Moreover, φ takes ω_1 many levels because for any $\alpha < \omega_1$, we can generate a 2-group of rank α by a well-founded tree of the same rank. Therefore R is strictly Π^1_1. ∎

Definition 4.2. A function f from a Polish space to another Polish space is partial Borel if it is the restriction of a Borel function on the domain of f.

Corollary 4.3. There is no partial Borel function $f : {}^{\omega}2 \to {}^{\omega}2$ such that if G is a 2-group then $f(G)$ is a maximal reduced subgroup of G.

Proof: If such a f exists then a 2-group G is reduced if and only if $f(G) = G$ according to our coding of subgroups. But this implies that the set $\{G : G$ is a reduced 2-group$\}$ is Borel, contradicting the previous theorem. ∎

Theorem 4.4. The set

$$\{(G_1; G_2) : G_1;\ G_2 \text{ are isomorphic reduced 2-groups}\} \tag{36}$$

is relatively Δ^1_1 in the set

$$\{(G_1; G_2) : G_1; G_2 \text{ are reduced 2-groups}\} \tag{37}$$

Proof: Since G_1 and G_2 are isomorphic if and only if there is an isomorphism between them, the above set is clearly relatively Σ^1_1.

It remains to prove that the set

$$\{(G_1; G_2) : G_1; G_2 \text{ are non-isomorphic reduced 2-groups}\} \tag{38}$$

is also relatively Σ^1_1.

By Ulm's theorem, G_1, G_2 are nonisomorphic if and only if one of the following is true:

(1) $rk(G_1) > rk(G_2)$

(2) $rk(G_2) > rk(G_1)$ or

(3) there exists $\beta < rk(G_1) = rk(G_2)$ such that $Ulm_{G_1}(\beta) \neq Ulm_{G_2}(\beta)$.

(1) is equivalent to the existence of a mapping $\varphi : T_{G_2} \to T_{G_1}$ such that φ preserves order in the trees, and the root of T_{G_2} is not mapped to the root of T_{G_1}: Therefore, it is a Σ^1_1 statement. Ditto for (2).
(3) can be rewritten as the following: there exists $x; f_1; f_2$ such that

x codes a countable ordinal and

$f_1 : x \to \omega$ codes the Ulm sequence of G_1
$f_2 : x \to \omega$ codes the Ulm sequence of G_2

and $f_1 \neq f_2$.

Since all of the above statements are Borel, the statement is proved. □

The following lemma provides an important tool for the proofs for most of the negative results in the rest of this chapter.

Lemma 4.5. For every Borel set $B \subseteq {}^\omega\omega$, there is an ordinal α such that for every $\beta \geq \alpha$; there is a continuous $f_\beta : B \to$ the set of well founded trees such that

$$x \in B \to f_\beta(x) \text{ has rank} < \alpha \tag{39}$$

$$x \notin B \to f_\beta(x) \text{ has rank } \beta \tag{40}$$

(More precisely, $f_\beta(x)$ codes a well founded tree $T_\beta(x)$, but we may identify these two from time to time.)

Proof: We may assume that the underlying sets of all our trees are subsets of the natural numbers with 0 being the root so that we can code a tree T by an element γ_T of the Cantor space such that:

$$\gamma_T(\langle i; i\rangle) = 1 \iff i \in T \tag{41}$$

and for $i \neq j$,

$$\gamma_T(\langle i; j\rangle) = 1 \iff i; j \in T \text{ and } i <_T j \tag{42}$$

Let B be a Borel set in the Baire space $\mathcal{N}$.
(i) B is clopen:

We may put $\alpha = 2$ and let T_0 be a rooted well-founded tree of rank 1. For each $\beta \geq \alpha$, we fix a rooted tree T_β of rank β and define a function $f_\beta : \mathcal{N} \to {}^\omega 2$ by

$$f_\beta(x) = \begin{cases} T_0 & \text{if } x \in B \\ T_\beta & \text{if } x \notin B \end{cases} \tag{43}$$

(ii) $B = \bigcap_{n \in \omega} B_n$:

We define $\alpha = \sup\{\alpha_n : n \in \omega\} + 1$ and for each $\beta \geq \alpha$ choose a sequence of continuous functions $\langle f_{n,\beta} : n \in \omega \rangle$ satisfying conditions

$$x \in B_n \rightarrow rk(f_{n,\beta}(x)) < \alpha_n \tag{44}$$
$$x \notin B_n \rightarrow rk(f_{n,\beta}(x)) = \beta \tag{45}$$

Define

$$T_\beta(x) = \text{the amalgamation of } \{T_{n,\beta}(x) : n \in \omega\} \text{ at the roots} \tag{46}$$

i.e. all $T_{n,\beta}(x)$'s share the same root and otherwise disjoint where $T_{n,\beta}(x)$ is the tree coded by $f_{n,\beta}(x)$.

Clearly $T_\beta(x)$ satisfies conditions (39), (40) and if we code it by an element in the Cantor space γ (this would be our $f_\beta(x)$) such that the underlying set of $T_{n,\beta}(x)$ is a subset of

$$\{0, \langle n, 1\rangle + 1, \langle n, 2\rangle + 1, \cdots\} \tag{47}$$

and for $i, j \neq 0$

$$\gamma(\langle\langle n,j\rangle+1, \langle n,j\rangle+1\rangle) = 1 \longleftrightarrow j \in T_{n,\beta}(x) \tag{48}$$
$$\gamma(\langle\langle n,i\rangle+1, \langle n,j\rangle+1\rangle) = 1 \longleftrightarrow i, j \in T_{n,\beta}(x) \text{ and } i < j \text{ in } T_{n,\beta}(x) \tag{49}$$

then any initial segment of γ will mention only a finite number of elements in a finite number of $T_{n,\beta}(x)$'s and since each $f_{n,\beta}$ is continuous, so is f_β.

(iii) $B = \bigcup_{n \in \omega} B_n$.

This time we let $\alpha = \sup\{\alpha_n : n \in \omega\} + \omega$ and again for each chosen countable ordinal $\beta \geq \alpha$, we choose a sequence of continuous functions $\langle f_{n,\beta} : n \in \omega \rangle$ as in the previous case.

For each $x \in \omega^\omega$ we define a tree $T_\beta(x)$ such that the m-th level of this tree consists of elements from the set

$$\{\langle t_0, t_1, \cdots, t_m\rangle : \forall i \leq m;\ t_i \text{ is on the m-th level of } f_{i,\beta}(x)\} \tag{50}$$

and define

$$\langle t_0, t_1, \cdots, t_m, t_{m+1}\rangle < \langle s_0, s_1, \cdots, s_m\rangle \longleftrightarrow t_i < s_i \text{ for all } i \leq m \tag{51}$$

Claim 1. For any $m \in \omega$, if $\langle t_0, t_1, \cdots, t_m\rangle \in T(x)$ and γ is an ordinal such that

(a) for all $n \in \omega$, $rk(f_{n,\beta}(x)) \geq \gamma + m$
(b) for all $i \leq m$, $rk(t_i) \geq \gamma$ in $f_{i,\beta}(x)$

then $rk(\langle t_0, t_1, \cdots, t_m\rangle) \geq \gamma$ in $T(x)$.

Proof: Induct on the ordinal γ. ⊣

Claim 2. For any $m \in \omega$, if $\langle t_0, t_1, \cdots, t_m\rangle \in T_\beta(x)$ and $rk(t_i) \leq \gamma$ in $T_{n,\beta}(x)$ for some $i \leq m$, then $rk(\langle t_0, t_1, \cdots, t_m\rangle) \leq \gamma$ in $T_\beta(x)$.

Proof: Suppose not, then we can project the subtree of $T_\beta(x)$ that consists of all elements at or below $\langle t_0, t_1, \cdots, t_m\rangle$ onto its i-th co-ordinate and that will give rise to a subtree of $T_{i,\beta}(x)$ whose rank is $> \gamma$. But this implies that $rk(t_i) > \gamma$ in $T_{i,\beta}$ which contradicts our assumption that $rk(t_i) \leq \gamma$ in $T_{i,\beta}(x)$. ⊣

Returning to the proof of the lemma, if $x \in B$ then $x \in B_i$ for some i and hence $f_{i,\beta}(x)$ has rank $< \alpha_i$ and in particular every element on the i-th level of $f_{i,\beta}(x)$ has rank $< \alpha_i$ hence by claim 2 so is every element on the i-th level of $T_\beta(x)$. This implies that the rank of $T_\beta(x)$ is at most $\alpha_i + i$ which is definitely less than α.

If $x \notin B$ then $f_{i,\beta}(x)$ has rank β for all $i \in \omega$ and by applying claim 1 and claim 2 to the element $\langle t_0\rangle$, where t_0 is the root of $T_{0,\beta}(x)$, we see that $rk(T_\beta(x))$ is exactly β.

Finally, let $f_\beta(x)$ be the element $\gamma \in {}^\omega 2$ that codes the tree $T_\beta(x)$ (we may assume that the elements in $T_\beta(x)$ are coded by the natural numbers in a recursive way) and by the same argument as in case (ii), we see that this f_β is continuous. □

Theorem 4.6. The set

$$\{(G, f) : G \text{ is a reduced 2-group and } f \text{ codes the Ulm invariant sequence of } G\} \tag{52}$$

is Σ^0_α-hard relative to the set $D = \{(G, f) : G$ is a reduced 2-group$\}$ (i.e. for every Σ^0_α set B, there is continous map whose range is contained in the set D and which reduces B to the above set) for every ordinal $\alpha < \omega_1$, hence it is not relatively Borel.

Proof: Let $A = \{(G, f) : G$ codes a reduced 2-group and f is the Ulm-sequence of $G\}$ and B be a Σ^0_α subset of the Baire space.

By lemma 4.5 there is an ordinal λ and a continuous function φ from ${}^\omega\omega$ to the set of countable well-founded trees such that

$$x \in B \quad \to \quad rk(\varphi(x)) < \lambda \tag{53}$$

$$x \notin B \quad \to \quad rk(\varphi(x)) = \lambda + 1 \tag{54}$$

Let $f : \lambda \to \{\aleph_0\}$ be the constant function with domain λ and H be a 2-group with rank λ and whose Ulm-sequence is f. Also, for each $x \in {}^\omega\omega$, let $K(x)$ be the 2-group generated by the well-founded tree $\varphi(x)$.

Finally we define $\psi : {}^\omega\omega \to {}^\omega 2 \times {}^\omega 2$ by

$$\psi(x) = (H \oplus K(x), f) \tag{55}$$

ψ is continuous because the map $x \mapsto K(x)$ is a composition of continuous maps.

It is not hard to see that ψ reduces B to A. □

Corollary 4.7. The set $\{(G_1; G_2) : G_1; G_2$ are isomorphic reduced 2-groups$\}$ is Σ^0_α-hard relative to the set $\{(G_1; G_2) : G_1; G_2$ are reduced 2-groups$\}$ for every ordinal $\alpha < \omega_1$, hence not relatively Borel.

Proof: Using the same notations as in the proof of the above theorem, let us define ψ' to be the continuous map

$$x \mapsto (H \oplus K(x); H) \tag{56}$$

According to the constuction of $K(x)$, $x \in B$ if and only if $H \oplus K(x)$ and H have the same rank and same Ulm-sequence. Therefore by Ulm's thoerem, we have $x \in B$ if and only if $H \oplus K(x)$ and H are isomorphic. $\square$

Theorem 4.4 and the above corollary imply that we have found a set in a Polish space which is relatively Δ^1_1 but not relatively Borel.

Theorem 4.8. There is no Borel partial function $f : {}^{\omega}2 \times {}^{\omega}2 \to {}^{\omega}2$ such that if G_1, G_2 are isomorphic countable reduced 2-groups, then $f(G_1; G_2)$ is an isomorhism between G_1 and G_2.

Proof: We shall see that if such a function f exists then there is a Borel way to determine whether $(rk_G(x) \geq rk_G(y)$ for any two arbitrary elements x, y in any reduced 2-group G, which is impossible by the next lemma.

We shall contruct a Borel partial map φ such that if G is a reduced 2-group then $\varphi(G)$ is a reduced 2-group satisfying

1. $rk(\varphi(G)) \geq rk(G)$.
2. for every $\alpha < rk(\varphi(G))$, the α-th Ulm invariant of $\varphi(G)$ is $\aleph_0$.
3. the set $D = \{(G; H; x; y) : G, H$ are reduced 2-groups, $H = \varphi(G)$, $x; y \in H$ and $rk_H(x) \geq rk_H(y)\}$ is relatively Borel.

Then since $G \oplus \varphi(G)$ and $\varphi(G)$ are isomorphic by Ulm's theorem, we can apply f to get an isomorphism $f_{(G\oplus\varphi(G);\varphi(G))}$ such that

$$\begin{aligned} rk_G(x) \geq rk_G(y) \quad &\leftrightarrow \quad rk_{G\oplus\varphi(G)}(\langle x; 0\rangle) \geq rk_{G\oplus\varphi(G)}(\langle y; 0\rangle) \\ &\leftrightarrow \quad rk_{\varphi(G)}\left(f_{(G\oplus\varphi(G);\varphi(G))}(\langle x; 0\rangle)\right) \geq rk_{\varphi(G)}\left(f_{(G\oplus\varphi(G);\varphi(G))}(\langle y; 0\rangle)\right) \end{aligned} \tag{57}$$

But then we can reduce the set $A = \{(G; x; y) : G$ is a reduced 2-group, $x; y \in G$ and $rk_G(x) \geq rk_G(y)\}$ to D by the Borel map

$$(G; x; y) \mapsto \left(G; \varphi(G); f_{(G\oplus\varphi(G);\varphi(G))}(\langle x; 0\rangle); f_{(G\oplus\varphi(G);\varphi(G))}(\langle y; 0\rangle)\right) \tag{58}$$

and this implies that A is also relatively Borel, a contradiction!

Now it remains to construct such a Borel partial map φ.

Let G be any reduced 2-group, we first redefine the full tree T_G of G to be a set of finite sequences of natural numbers such that

$$
\begin{aligned}
(n_0; n_1; \cdots; n_k) \in T_G \quad \leftrightarrow \quad & n_0; n_1; \cdots; n_k \in G \\
& n_0 \text{ is the identity and} \\
& \forall i < k;\ 2n_{i+1} = n_i
\end{aligned}
\tag{59}
$$

and these finite sequences are ordered by extension, i.e. $\bar{u} < \bar{v}$ if and only if $\bar{u}$ extends $\bar{v}$.

Let $<^*$ be the Kleene-Brouwer ordering on T_G based on the standard ordering of ω; namely

$$
\begin{aligned}
(v_0; \cdots; v_s) <^* (u_0; \cdots; u_t) \quad \leftrightarrow \quad & (v_0; \cdots; v_s); (u_0; \cdots; u_t) \in T_G \text{ and} \\
& \{[v_0 < u_0] \vee [v_0 = u_0 \ \& \ v_1 < u_1] \vee \\
& [v_0 = u_0 \ \& \ v_1 = u_1 \ \& \ v_2 < u_2] \vee \\
& \cdots\cdots \\
& \vee [v_0 = u_0 \ \& \ v_1 = u_1 \& \cdots \ \& \ v_t = u_t \ \& \ s > t]\}
\end{aligned}
$$

and this linear ordering will then induce an ordering $<^{**}$ on G in the following manner,

$$
\begin{aligned}
y <^{**} x \quad \leftrightarrow \quad & (v_0; \cdots; v_s; y) <^* (u_0; \cdots; u_t; x) \\
& \text{where } \bar{u}, \bar{v} \text{ are the unique sequences} \\
& \text{such that } \bar{u}^\frown\{y\} \text{ and } \bar{v}^\frown\{x\} \text{ belong to } T_G
\end{aligned}
\tag{60}
$$

The relation $\{(G; x; y) : G$ is a reduced 2-group $; x; y \in G$ and $x <^{**} y\}$ is then clearly relatively Borel.

If G is reduced then T_G is well-founded and $<^*$, $<^{**}$ will be well orderings on T_G and G respectively. Moreover, the order type $\alpha(G)$ of $<^{**}$ is no less than the rank of G because the tree ordering is embeddable into the linear order $<^{**}$.

Next we shall build a tree $T_U(G)$ of rank $\alpha(G)$ such that every Ulm invariant of the 2-group generated by $T_U(G)$ is $\aleph_0$.

Let $T_U(G)$ be the amalgamation at the root of $\aleph_0$ copies of the tree $T_0(G)$ whose underlying set is the set of all finite sequences of natural numbers $\langle n_0; n_1; \cdots; n_k \rangle$ such that

$n_0; n_1; \cdots; n_k \in G$ and $n_0 {}^{**}> n_1 {}^{**}> n_2 {}^{**}> \cdots {}^{**}> n_k$.

and these sequences are ordered by extension with the empty sequence being the root.

It is easy to prove by induction that the rank of any $\langle n_0; \cdots; n_k \rangle$ in $T_0(G)$ is just the order type of $\mathrm{pred}(G; <^{**}; n_k)$. Hence

$$
\text{“}rk_{T_0(G)}(\langle n_0; \cdots; n_k \rangle) < rk_{T_0(G)}(\langle m_0; \cdots; m_l \rangle) \longleftrightarrow n_k <^{**} m_l\text{”}
\tag{61}
$$

Every Ulm invariant of $T_0(G)$ is at least 1 because for every $\beta < \alpha(G)$, we can find an element $n \in G$ such that o.t.$(\mathrm{pred}(G; <^{**}; n)) = \beta$ and hence $rk_{T_0}(\langle n \rangle) = \beta$. But the element $\langle n \rangle$ in the group generated by $T_0(G)$ has order 2, so the β-th Ulm invariant of $T_0(G)$ is non-zero and that of $T_U(G)$ will be $\aleph_0$.

Finally let φ be the map "$G \mapsto$ the 2-group generated by $T_U(G)$" □

Lemma 4.9. The set

$$\{(G; x; y) : G \text{ is a countable reduced 2-group, } x; y \in G \text{ and } rk_G(x) \ge rk_G(y)\} \quad (62)$$

is not relatively Borel in the set

$$\{(G; x; y) : G \text{ is a countable reduced 2-group, } x; y \in G\} \quad (63)$$

Proof: We shall show that every Borel subset of the Baire space can be reduced to this given set. Let $B \subseteq \omega^\omega$ be any chosen Borel set. By Lemma 4.5 there is an $\alpha \in \omega_1$ such that for every $\beta \ge \alpha$ there is a continuous map

$$f_\beta : B \rightarrow \text{the set of well-founded trees} \quad (64)$$

such that

$$x \in B \rightarrow f_\beta(x) \text{ has rank } < \alpha \quad (65)$$

$$x \notin B \rightarrow f_\beta(x) \text{ has rank } \beta \quad (66)$$

To be specific, let $\beta = \alpha + 1$ and without loss of generality we may assume that, for all x, the underlying set of $f_\beta(x)$ is a subset of the even numbers greater than 0 and 2 is its root.

Let T_0 be a fixed well-founded tree of rank α such that its underlying set is a subset of the odd numbers and 1 is its root.

Now we can define, for each x, a tree T_x as illustrated in Figure 4.

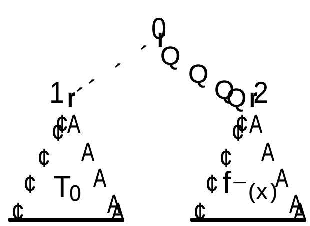


Figure 3:

and define G_x to be the reduced 2-group generated by T_x.

Finally if we define φ to be the map

$$x \mapsto (G_x; 1; 2) \tag{67}$$

then $rk_{G_x}(1) > rk_{G_x}(2)$ if and only if $x \in B$, and hence φ is a continuous map (because f^{-} is) reducing B to the given set. $\square$

Remark: The above construction in fact proves a stronger version of the lemma, namely,

"$\{(G; x; y) : G$ is a countable reduced 2-group, $x; y \in G$, $o(x) = o(y) = 2$ and $rk_G(x) \geq rk_G(y)\}$ is not relatively Borel".

Theorem 4.10. There is no Borel function $f : {}^{\omega}2 \times {}^{<\omega}2 \to \{0; 1\}$ such that if G is a countable reduced 2-group and T is a finite subtree of the full tree T_G, then

$$f(G; T) = 1 \longleftrightarrow T \text{ can be extended to a generating tree of } G \tag{68}$$

Proof: We shall show that if such a Borel function exists, then the set

$$\begin{aligned}\{(G; x; y) \ :\ & G \text{ is a reduced 2-group, } x; y \in G; \\ & o(x) = o(y) = 2 \text{ and } rk_G(x) \geq rk_G(y)\}\end{aligned} \tag{69}$$

is relatively Borel, which contradicts the above remark.

Given x, y in G, note that

$$\begin{aligned} rk(x) < rk(y) \quad \leftrightarrow \quad & [rk(x) = 0; \ rk(y) > 0] \text{ or} \\ & [rk(x) > 0; \ rk(y) > 0 \text{ and} \\ & rk(x) < rk(y) \end{aligned} \tag{70}$$

The first condition on the right hand side of the above equivalence is clearly relatively Borel, so we only need to take care of the second condition.

If $rk(y) > 0$ but $rk(y) \leq rk(x)$ then for every z below y, the following finite tree $T_{(x;y;z)}$ (Figure 5) cannot be extended to a generating tree for G because it is not rank independent.

It is not hard to prove that if $rk(y) > 0$, then $rk(y) > rk(x)$ if and only if there exists a z directly below y such that the tree $T_{(x;y;z)}$ can be extended to a generating tree for G.

Now the relation

$$rk(y) > rk(x) \tag{71}$$

can be rewritten as

$$\exists z \ 2z = y \ \wedge \ f(G; T_{(x;y;z)}) = 1 \tag{72}$$

which is a relatively Borel relation.

By corollary 3.8, we know that G has at least one nice generating tree, say T. We shall then modify T into a generating tree $\tilde{T}$ of G which contains $T_{(x;y;z)}$ as a subtree.

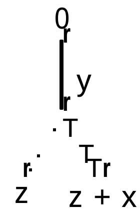


Figure 4: $T_{(x;y;z)}$

Step 1: Change T to a generating tree T' that contains y.

If $y \in T$, let $T' = T$; otherwise proceed as follows. Let $a_1; a_2; \cdots; a_k \in T \setminus \{0\}$ such that

$$y = a_1 + \cdots + a_k \quad (73)$$

with

$$rk(y) = rk(a_1) \leq rk(a_2) \leq \cdots \leq rk(a_k) \quad (74)$$

T' will be the amalgamation of T'_1 and T'_2 at the root where

$$T'_1 = T \setminus \{t \in T : t \leq a_1\} \quad (75)$$

and T'_2 will be a tree isomorphic to the subtree of T defined by

$$\{t \in T : t = 0 \text{ or } t \leq a_1\} \quad (76)$$

Let us construct T'_2 level by level.
0th level: $\{0\}$
1st level: $\{y\}$
2nd level:

For each $b_1 \in T$ directly below a_1, we pick the first b_i directly below a_i such that $rk(b_i) \geq rk(b_1)$ for all $i = 2; \cdots; k$. This is possible because $rk(a_i) \geq rk(a_1)$ for all $i = 2; \cdots; k$. Since $2(b_1 + \cdots + b_k) = y$ we can put $b_1 + \cdots + b_k$ into T'_2 directly below y; and it is clear that $rk(b_1 + \cdots + b_k) = rk(b_1)$.
3rd level:

Similar to level 2, if $d_1 + \cdots + d_k$ is on the 2nd level of T_2 with d_1 directly below a_1 and there is $c_1 \in T$ directly below d_1, we then pick the first $c_i \in T$ directly below d_i with $rk(c_i) \geq rk(c_1)$ for all $i = 2; \cdots; k$, and put $c_1 + \cdots + c_k$ into T_2 directly below $d_1 + \cdots + d_k$.

All the lower levels will be constructed in a similar way.
Step 2: Change T' to $\tilde{T}$
Case (i) $x \in T'$:

Let $T'' = T' \setminus \{t \in T' : t \leq x\}$ and pick a $z \in T'$ directly below y such that $rk(z) \geq rk(x)$.

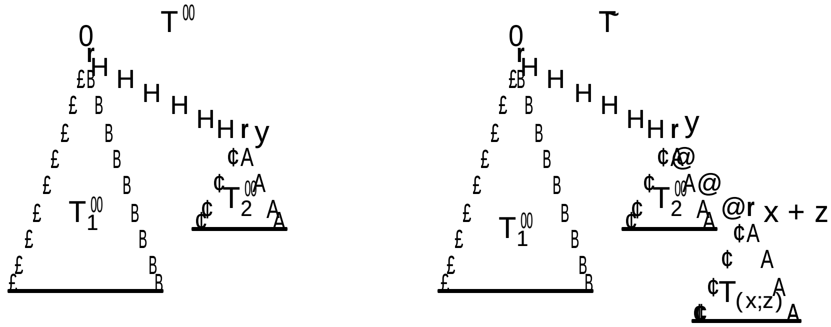


Figure 5:

We then construct a tree $T_{(x;z)}$ which is isomorphic to the subtree

$$\{t \in T' : t \le x\} \tag{77}$$

using a method similar to the construction of T_2''.

Our $\tilde{T}$ would then be $T'' \cup T_{(x;z)}$ with $x + z$ attached directly below y (as shown in Figure 6).

Case (ii) $x \notin T'$:

We can use the method in step 1 to modify T' to a generating tree T'' which contains x. And in this case, because $rk(y) > rk(x)$, y would still be in T'' and we are back to case (i).

□

Theorem 4.11. For any fixed $\alpha < \omega_1$, there is a Borel way to get a nice generating tree for any reduced 2-group G of rank less than α.

Let us first extend the definition of the rank of an element in any Abelian p-group G:

For any $g_0 \in G$, if the subtree $\{g \in G : g \le g_0\}$ of the full tree T_G is well-founded, then $rk_G(g_0)$ is defined to be the rank of this subtree, otherwise the rank of g_0 is defined to be ∞.

Lemma 4.12. For every ordinal $\alpha < \omega_1$, there is a Borel function $\Phi_\alpha : {}^{\omega}2 \to {}^{\omega}2$ defined on the set of countable Abelian 2-groups such that $\Phi_\alpha(G)$ is a real number coding the function $f_G : G \to \alpha \cup \{\infty\}$ satisfying the following conditions:

(1) If G is reduced and $rk(G) \le \alpha$ then $\forall g \in G \setminus \{0\}$, $f_G(g) = rk_G(g)$.

(2) If $rk(G) > \alpha$ or G is not reduced, then

$$f_G(g) = \begin{cases} \infty & \text{if } rk_G(g) \text{ is undefined or } \ge \alpha \\ rk_G(g) & \text{otherwise} \end{cases} \tag{78}$$

Proof: For each countable ordinal ®, let us …x a recursive bijective map

¡® : ® [f 1 g ! ! (or a …nite subset of !) (79)

which codes the ordinal ® and the symbol 1 :

We then proceed by induction on ®.

(1) If ® = 0, all the functions ©0(G) are constant functions and hence ©0 is Borel.

(2) If ® = ¯ + 1 is a successor, let ©¯ be the Borel function with the desired properties. We then de…ne

©®(G)(g) = 8 < : ¡® ± ¡¯⁻¹(©¯(G)(g)) if ©¯(G)(g) 6= ¡¯(1) ; ¡®(¯) if ©¯(G)(g) = ¡¯(1); but for every h directly below g; ©¯(G)(g) 6= ¡¯(1) ; ¡®(1) otherwise (80)

(3) When ® is a limit, we …x a co…nal sequence ®1; ®2; ¢¢¢ of ordinals and de…ne

©®(G)(g) = 8 < : ¡® ± ¡®n⁻¹(©®n(G)(g)) where n is the smallest such that ©®n(G)(g) 6= ¡®n(1), if such n exists ; ¡®(1) otherwise (81)

2

Lemma 4.13. For every ® < !1, there is a Borel function ª® whose domain is the set of countable Abelian 2-groups. If G is reduced and of rank **6** ® then ª®(G) is the Ulm sequence of G, otherwise it is the identity function.

More precisely, if G is of rank **6** ®;

ª®(G) : ! ! ! (82)

is a function such that 8¯ < ®,

ª®(G)(¡®(¯)) = 0 $ UlmG(¯) = @0 (83)

ª®(G)(¡®(¯)) = k + 1 $ UlmG(¯) = k (84)

ª®(G)(¡®(1)) = 0 (85)

where ¡® : ® [f 1 g ! ! (or a …nite subset of !) is the same recursive bijection as in the previous lemma.

Proof:

Since the domain of ª® is Borel, it su¢ces to prove that its graph is Borel.

Let ©® be the Borel function de…ned in the previous lemma. Then for any given G with rk(G) · ®, ©®(G) will be the rank function of G.

For every G and $h : \omega \to \omega$, $\Psi_\alpha(G) = h$ if and only if the following is true:

$$G \in dom(\Psi_\alpha) \wedge \exists f\, [f = \Phi_\alpha(G) \wedge \forall m \forall k, \; m \in range(\Gamma_\alpha) \wedge h(m) = k \to \{k > 0 \wedge \varphi(k-1, m) \wedge \neg\varphi(k, m)\} \vee \{k = 0 \wedge \forall n\, \varphi(n, m)\}] \tag{86}$$

where $\varphi(k, m)$ is the sentence:

"G has k distinct elements $g_1, \cdots, g_k$ such that

$$o(g_1) = \cdots = o(g_k) = 2 \quad \& \quad f(g_1) = \cdots = f(g_k) = m \tag{87}$$

(i.e. their ranks are all equal to the ordinal coded by m)

and $g_1, \cdots, g_k$ are rank independent."

Similarly, the relation $\Psi_\alpha(G) = h$ can also be expressed by a Π^1_1 formular, and therefore it is a Borel relation.

$\Box$

Now we can proceed to prove theorem 4.11.

Proof: (of theorem 4.11) The construction consists of several steps.

Let G be any given Abelian reduced 2-group with rank $\leq \theta$.

(1) Obtain the Ulm sequence of G by the function constructed in the above lemma.

(2) Build a nice tree T such that the 2-group H generated by T has the same Ulm sequence as G.

(3) By Ulm's theorem, G and H are isomorphic and hence there is an isomorphism $\varphi : H \to G$. In addition, according to the proof of Ulm's theorem (see [6]), the isomorphism is constructed in a back and forth process in which an element of certain order and certain rank is chosen in each step. For groups of bounded ranks, this is a Borel process as a consequence of lemma 4.12.

(4) Obtain the image of T under φ which will then be a nice generating tree for G. $\Box$

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